

# CRYSTALS, QUIVER VARIETIES, AND COBOUNDARY CATEGORIES FOR KAC-MOODY ALGEBRAS

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**ABSTRACT.** Henriques and Kamnitzer have defined a commutor for the category of crystals of a finite-dimensional complex reductive Lie algebra that gives it the structure of a coboundary category (somewhat analogous to a braided monoidal category). Kamnitzer and Tingley then gave an alternative definition of the crystal commutor, using Kashiwara's involution on Verma crystals, that generalizes to the setting of symmetrizable Kac-Moody algebras. In the current paper, we give a geometric interpretation of the crystal commutor using quiver varieties. Equipped with this interpretation we show that the commutor endows the category of crystals of a symmetrizable Kac-Moody algebra with the structure of a coboundary category, answering in the affirmative a question of Kamnitzer and Tingley.

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## INTRODUCTION

Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra and  $U_q(\mathfrak{g})$  the corresponding quantum group (or quantized universal enveloping algebra). Introduced by Kashiwara, crystals can be thought of as a combinatorial model of representations of  $U_q(\mathfrak{g})$  arising from the limit as  $q$  tends to zero. To each representation of  $U_q(\mathfrak{g})$  is associated a crystal graph. Roughly speaking, the crystal graph is an edge-colored directed graph in which a certain basis (the global, or canonical, basis) of the representation is replaced by a set of vertices and the action of the Chevalley generators is replaced by colored arrows. Arrows are labeled by simple roots of  $\mathfrak{g}$  and to each vertex is associated a weight of  $\mathfrak{g}$ . One can take the tensor product of two crystals and this operation corresponds to the tensor product of the corresponding representations. The vertex set of the tensor product crystal is the Cartesian product of the two original vertex sets. With this operation, the category of  $\mathfrak{g}$ -crystals becomes a monoidal category. As for representations of  $U_q(\mathfrak{g})$ , the tensor product of crystals is not symmetric. That is, the map  $(b_1, b_2) \mapsto (b_2, b_1)$  is not a morphism of crystals in general.

Recall that a braided monoidal category is a monoidal category  $\mathcal{C}$  equipped with a natural isomorphism  $\sigma_{V,W}^{br} : V \otimes W \rightarrow W \otimes V$  for all  $V, W \in \text{Ob } \mathcal{C}$  and such that the diagram

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{\text{Id} \otimes \sigma_{V,W}^{br}} & U \otimes W \otimes V \\ & \searrow \sigma_{U \otimes V, W}^{br} & \downarrow \sigma_{U,W}^{br} \otimes \text{Id} \\ & & W \otimes U \otimes V \end{array}$$

commutes for all  $U, V, W \in \text{Ob } \mathcal{C}$ . Such a  $\sigma^{br}$  is called a *braiding* and it induces an action of the braid group on multiple tensor products. We refer the reader to [1, §5.2] for further details.

For  $\mathfrak{g}$  a finite-dimensional complex Lie algebra, the category of representations of  $U_q(\mathfrak{g})$  has a natural braiding constructed using the universal  $R$ -matrix, an element of  $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$  where  $U_h(\mathfrak{g})$  is the formal completion of  $U_q(\mathfrak{g})$ . The braiding is given by the map  $\text{flip} \circ R$  where  $\text{flip} : V \otimes W \rightarrow W \otimes V$  is given by  $v \otimes w \mapsto w \otimes v$ . However, this braiding does not pass to the  $q \rightarrow 0$  limit. In other words, it does not induce a braiding on the category of crystals. In fact, one can show that no such braiding exists. That is, the category of  $\mathfrak{g}$ -crystals cannot be given the structure of a braided monoidal category for nontrivial  $\mathfrak{g}$ . However, it can be given an analogous structure as we now describe.

A *coboundary* (or *cactus*) category is a monoidal category  $\mathcal{C}$  equipped with a natural isomorphism  $\sigma_{V,W} : V \otimes W \rightarrow W \otimes V$  for all  $V, W \in \text{Ob } \mathcal{C}$ , called a *commutor*, such that  $\sigma_{W,V} \circ \sigma_{V,W} = \text{Id}$  and the diagram

$$(0.1) \quad \begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{\text{Id} \otimes \sigma_{V,W}} & U \otimes W \otimes V \\ \sigma_{U,V} \otimes \text{Id} \downarrow & & \downarrow \sigma_{U,W \otimes V} \\ V \otimes U \otimes W & \xrightarrow{\sigma_{V \otimes U, W}} & W \otimes V \otimes U \end{array}$$

commutes for all  $U, V, W \in \text{Ob } \mathcal{C}$ . The commutativity of (0.1) is called the *cactus relation*. A commutor satisfying these conditions induces an action of the *cactus group* (see [3]) on multiple tensor products.

Drinfel'd [2] has shown that one can use the  $R$ -matrix to construct a commutor satisfying the cactus relation in the category  $U_q(\mathfrak{g})$  via a process he calls *unitarization*. If one defines  $R' = R(\text{flip}(R)R)^{-1/2}$  where the square root is with respect to the  $h$  filtration on  $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ , then the map  $\text{flip} \circ R'$  is a commutor. In [3], Henriques and Kamnitzer, following an idea of A. Berenstein, defined a commutor on the category of representations of  $U_q(\mathfrak{g})$  and the category of  $\mathfrak{g}$ -crystals for  $\mathfrak{g}$  a finite-dimensional complex reductive Lie algebra. Their definition involved the Schützenberger involution which only exists in finite type. It was shown by Kamnitzer and Tingley in [6] that a particular case of Henriques and Kamnitzer's construction agrees with Drinfel'd's commutor and that this unitarization does pass to the  $q \rightarrow 0$  limit. That is, it induces the structure of a coboundary category on the category of  $\mathfrak{g}$ -crystals for  $\mathfrak{g}$  of finite type. In [7], Kamnitzer and Tingley gave an alternative definition of the commutor using Kashiwara's involution. The new approach has the benefit of defining the involution  $\sigma$  for arbitrary symmetrizable Kac-Moody algebras. Thus, there exist two combinatorial definitions of the commutor: the definition of [3] where the cactus relation can easily be seen to hold but which does not generalize to the Kac-Moody setting, and the definition of [7] which does generalize to the Kac-Moody setting but for which the cactus relation has not been shown to hold (for non-finite type). Kamnitzer and Tingley posed the natural question of whether or not the commutor, extended to symmetrizable Kac-Moody algebras via the second definition, satisfies the cactus relation in the more general setting.

The goal of the current paper is twofold. First, we give a geometric interpretation of the crystal commutor using the quiver varieties of Lusztig and Nakajima. These are varieties associated to quivers (directed graphs) constructed from the Dynkin graph of a symmetrizable Kac-Moody algebra  $\mathfrak{g}$ . The set of irreducible components of these varieties can be given the structure of a  $\mathfrak{g}$ -crystal in a natural geometric way. We give a geometric characterization of these irreducible components and use this description to analyze the action of the crystal commutor described above. In doing so, we attain our second goal. Namely, we answer the above question of Kamnitzer and Tingley in the affirmative: the crystal commutor satisfies the cactus relation for an arbitrary symmetrizable Kac-Moody algebra  $\mathfrak{g}$  and therefore endows the category of  $\mathfrak{g}$ -crystals with the structure of a coboundary category. The key ingredient in the proof is that in the language of quiver varieties, the two compositions of commutors appearing in the cactus relation (0.1) both correspond to taking adjoints of quiver representations (at least when we restrict them to highest weight elements) and are therefore equal.

The paper is organized as follows. In Section 1 we review the definition of the crystal commutor using Kashiwara's involution. In Section 2 we introduce the quiver varieties of Lusztig, Malkin and Nakajima. The geometric realization of the crystals corresponding to the lower half of the quantized enveloping algebra of a symmetric Kac-Moody algebra and its integrable highest weight representations is given in Section 3. In Section 4 we discuss various characterizations of the irreducible components of quiver varieties and examine how the crystal commutor acts on their irreducible components. Equipped with a precise description of this action, we prove that the commutor satisfies the cactus relation in Section 5. Finally, in Section 6, we extend our results to the case of Kac-Moody algebras with symmetrizable (rather than symmetric) Cartan matrices by a well-known “folding” argument.

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## 1. CRYSTALS AND COBOUNDARY CATEGORIES

In this section we introduce the crystal commutor as defined by Kamnitzer and Tingley in [7]. It was defined in a different manner for the case of finite-dimensional complex reductive Lie algebras by Henriques and Kamnitzer in [3]. We refer the reader to [20] for a more detailed overview of the topic. In the current paper, by the *category of  $\mathfrak{g}$ -crystals* for a symmetrizable Kac-Moody algebra  $\mathfrak{g}$ , we mean the category consisting of those crystals  $B$  such that each connected component of  $B$  is isomorphic to some  $B_\lambda$ , the crystal corresponding to the irreducible highest weight  $U_q(\mathfrak{g})$ -module of highest weight  $\lambda$ , where  $\lambda$  is a dominant integral weight. For the rest of this paper, the word *crystal* means either an object in this category or the crystal  $B_\infty$  corresponding to the lower half  $U_q(\mathfrak{g})^-$  of the quantized universal enveloping algebra.

**1.1. Kashiwara's involution.** Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra and let  $B_\infty$  be the  $\mathfrak{g}$ -crystal corresponding to the lower half  $U_q(\mathfrak{g})^-$  of the quantized universal enveloping algebra. Let  $*$  :  $U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$  be the anti-automorphism given by

$$\begin{aligned} q^* &= q, \\ e_i^* &= e_i, \\ f_i^* &= f_i, \\ q(h)^* &= q(-h). \end{aligned}$$

The map  $*$  sends  $U_q(\mathfrak{g})^-$  to  $U_q(\mathfrak{g})$  and induces a map  $*$  :  $B_\infty \rightarrow B_\infty$  (see [8, §8.3]). Setting

$$\begin{aligned} \tilde{e}_i^*(b) &= (\tilde{e}_i(b^*))^*, \\ \tilde{f}_i^*(b) &= (\tilde{f}_i(b^*))^*, \\ \tilde{\varepsilon}_i^*(b) &= \varepsilon_i(b^*), \\ \tilde{\varphi}_i^*(b) &= \varphi_i(b^*), \end{aligned}$$

gives  $B_\infty$  another crystal structure. We call the map  $*$  *Kashiwara's involution*.

Let  $B_\lambda$  be the  $\mathfrak{g}$ -crystal corresponding to the irreducible highest weight  $U_q(\mathfrak{g})$ -module of highest weight  $\lambda$  and let  $b_\lambda$  be its highest weight element. We recall the tensor product rule for crystals.

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}, \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}, \\ \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle). \end{aligned}$$

For two dominant integral weights  $\lambda$  and  $\mu$ , there is an inclusion of crystals  $B_{\lambda+\mu} \hookrightarrow B_\lambda \otimes B_\mu$  sending  $b_{\lambda+\mu}$  to  $b_\lambda \otimes b_\mu$ . It follows from the tensor product rule that the image of this inclusion contains all elements of the form  $b \otimes b_\mu$  for  $b \in B_\lambda$ . Thus we define a map

$$\iota_\lambda^{\lambda+\mu} : B_\lambda \rightarrow B_{\lambda+\mu}$$

which sends  $b \in B_\lambda$  to the inverse image of  $b \otimes b_\mu$  under the inclusion  $B_{\lambda+\mu} \hookrightarrow B_\lambda \otimes B_\mu$ . While this map is not a morphism of crystals, it takes  $b_\lambda$  to  $b_{\lambda+\mu}$  and is  $\tilde{e}_i$ -equivariant. Here  $\tilde{e}_i$ -equivariant means that  $\iota_\lambda^{\lambda+\mu}(\tilde{e}_i b) = \tilde{e}_i \iota_\lambda^{\lambda+\mu}(b)$  for all  $i$  (note that it follows that  $\tilde{e}_i b = 0$  whenever  $\tilde{e}_i \iota_\lambda^{\lambda+\mu}(b) = 0$ ). This notion of  $\tilde{e}_i$ -equivariant is sometimes called  $\tilde{e}_i$ -strict

The maps  $\iota_\lambda^{\lambda+\mu}$  make the family of crystals  $B_\lambda$  into a directed system and the crystal  $B_\infty$  can be viewed as the limit of this system. We have  $\tilde{e}_i$ -equivariant maps  $\iota_\lambda^\infty : B_\lambda \rightarrow B_\infty$  which we will simply denote by  $\iota^\infty$  when it will cause no confusion. We define  $\iota^\infty : B \rightarrow B_\infty$  for an arbitrary  $\mathfrak{g}$ -crystal  $B$  by setting  $\iota^\infty(b) = \iota_\lambda^\infty(b)$  if the connected component of  $B$  containing  $b$  is isomorphic to  $B_\lambda$ . This extended map  $\iota^\infty$  is also  $\tilde{e}_i$ -equivariant. Define  $\varepsilon^* : B_\infty \rightarrow P_+$  by

$$\varepsilon^*(b) = \min\{\lambda \mid b \in \iota^\infty(B_\lambda)\}$$

where we put the usual order on  $P_+$ , the positive weight lattice of  $\mathfrak{g}$ , given by  $\lambda \geq \mu$  if and only if  $\lambda - \mu \in P_+$ . Recall that we also have the map  $\varepsilon : B_\infty \rightarrow P_+$  given by  $\langle \alpha_i^\vee, \varepsilon(b) \rangle = \varepsilon_i(b)$ . Then by [8, Prop. 8.2], Kashiwara's involution preserves weights and satisfies

$$(1.1) \quad \varepsilon^*(b) = \varepsilon(b^*).$$

**1.2. The crystal commutor.** Consider the crystal  $B_\lambda \otimes B_\mu$ . Since  $\varphi(b) = \varepsilon(b) + \text{wt}(b)$  for all  $b \in B_\lambda$ , we have that  $\varphi(b_\lambda) = \text{wt}(b_\lambda) = \lambda$ . It follows from the tensor product rule for crystals that the highest weight elements of  $B_\lambda \otimes B_\mu$  are those elements of the form  $b_\lambda \otimes b$  for  $b \in B_\mu$  with  $\varepsilon(b) \leq \lambda$ . Thus  $\varepsilon^*(b^*) = \varepsilon(b) \leq \lambda$  and so, by the definition of  $\varepsilon^*$ , we have  $b^* \in \iota^\infty(B_\lambda)$ . So we can consider  $b^*$  as an element of  $B_\lambda$ . Furthermore,  $\varepsilon(b^*) = \varepsilon^*(b) \leq \mu = \varphi(b_\mu)$  since  $b \in B_\mu$ . Thus  $b_\mu \otimes b^*$  is a highest weight element of  $B_\mu \otimes B_\lambda$ . Since  $B_\lambda \otimes B_\mu \cong B_\mu \otimes B_\lambda$  as crystals, we can make the following definition.

**Definition 1.1** ([7]). *Let  $\sigma_{B_\lambda, B_\mu} : B_\lambda \otimes B_\mu \xrightarrow{\cong} B_\mu \otimes B_\lambda$  be the crystal isomorphism given uniquely by  $\sigma_{B_\lambda, B_\mu}(b_\lambda \otimes b) = b_\mu \otimes b^*$  for  $b_\lambda \otimes b$  a highest weight element of  $B_\lambda \otimes B_\mu$ . The map  $\sigma_{B_\lambda, B_\mu}$  is called the crystal commutor.*

Note that it is enough to define the crystal commutor  $\sigma_{B_1, B_2} : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$  when  $B_1$  and  $B_2$  are highest weight crystals since all objects in the category of  $\mathfrak{g}$ -crystals are unions of these by definition. It was shown in [3, 7] that for  $\mathfrak{g}$  a finite-dimensional complex reductive Lie algebra, the commutor satisfies the cactus relations (0.1) and thus endows the category of  $\mathfrak{g}$ -crystals with the structure of a coboundary (or cactus) category. One of the goals of the current paper is to show that this is true for  $\mathfrak{g}$  an arbitrary symmetrizable Kac-Moody algebra.

## 2. QUIVER VARIETIES

In this section we introduce the quiver varieties of Lusztig and Nakajima and the tensor product varieties defined by Malkin and Nakajima.

**2.1. Lusztig quiver varieties.** Let  $I$  be the set of vertices of the Dynkin graph of a Kac-Moody Lie algebra  $\mathfrak{g}$  with symmetric Cartan matrix and let  $H$  be the set of pairs consisting of an edge together with an orientation of it. We call the elements of  $H$  *arrows*. Denote the corresponding quiver by  $Q = (I, H)$ . For  $h \in H$ , let  $\text{in}(h)$  (resp.  $\text{out}(h)$ ) be the incoming or tip (resp. outgoing or tail) vertex of  $h$ . We define the involution  $\bar{\cdot} : H \rightarrow H$  to be the function which takes  $h \in H$  to the element of  $H$  consisting of the same edge with opposite orientation. An *orientation* of our graph is a choice of a subset  $\Omega \subset H$  such that  $\Omega \cup \bar{\Omega} = H$  and  $\Omega \cap \bar{\Omega} = \emptyset$ . A *directed path* in  $Q$  is a sequence  $h_k \dots h_2 h_1$  where  $h_i \in H$  for  $1 \leq i \leq k$  and  $\text{out}(h_{i+1}) = \text{in}(h_i)$  for  $1 \leq i \leq k-1$ . The *length* of such a path is  $k$ .

Let  $\mathcal{V}$  be the category of finite-dimensional  $I$ -graded vector spaces  $V = \bigoplus_{i \in I} V_i$  over  $\mathbb{C}$  with morphisms being linear maps respecting the grading. Then  $V \in \mathcal{V}$  shall denote that  $V$  is an object of  $\mathcal{V}$ .

Given  $V^1, V^2 \in \mathcal{V}$ , let

$$E(V^1, V^2) = \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}^1, V_{\text{in}(h)}^2)$$

$$L(V^1, V^2) = \bigoplus_{i \in I} \text{Hom}(V_i^1, V_i^2).$$

For  $x = (x_h) \in E(V^1, V^2)$  and  $y = (y_h) \in E(V^2, V^3)$ , define  $yx \in L(V^1, V^3)$  to be the element with  $i$ th component

$$\sum_{h \in H, \text{in}(h)=i} y_h x_{\bar{h}}$$

The products  $ts, ys, tx$  for  $s \in L(V^1, V^2)$  and  $t \in L(V^2, V^3)$  are defined in the obvious way. For  $s \in L(V, V)$ , we define  $\text{tr } a = \sum_{i \in I} \text{tr}(a_i)$ .

The algebraic group  $G_V = \prod_i GL(V_i)$  acts on  $E(V, V)$  by  $g \cdot x = gxg^{-1}$ . The Lie algebra of  $G_V$  is  $\mathfrak{gl}_V = \prod_i \text{End}(V_i)$ .

Define the function  $\epsilon : H \rightarrow \{-1, 1\}$  by

$$\epsilon(h) = \begin{cases} 1 & \text{if } h \in \Omega \\ -1 & \text{if } h \in \bar{\Omega} \end{cases}.$$

For  $x \in E(V^1, V^2)$ , define  $\epsilon x \in E(V^1, V^2)$  by  $(\epsilon x)_h = \epsilon(h)x_h$  for  $h \in H$ . Then let  $\langle \cdot, \cdot \rangle$  be the nondegenerate,  $G_V$ -invariant, symplectic form on  $E(V, V)$  with values in  $\mathbb{C}$  defined by

$$\langle x, y \rangle = \text{tr}((\epsilon x)y).$$

The moment map associated to the  $G_V$ -action on the symplectic vector space  $E(V, V)$  is the map  $\psi : E(V, V) \rightarrow \mathfrak{gl}_V$  given by

$$\psi(x) = (\epsilon x)x.$$

Here we have identified  $\mathfrak{gl}_V$  with its dual via the trace.

**Definition 2.1.** An element  $x \in E(V, V)$  is said to be nilpotent if there exists an  $N \geq 1$  such that for any directed path  $h_N \dots h_2 h_1$  of length  $N$ , the composition  $x_{h_N} \dots x_{h_2} x_{h_1} : V_{\text{out}(h_1)} \rightarrow V_{\text{in}(h_N)}$  is zero.

Let  $\Lambda(V)$  be the set of all nilpotent elements  $x \in E(V, V)$  such that  $\psi(x) = 0$ . Since, up to isomorphism, it depends only on the graded dimension  $\mathbf{v}$  of  $V$ , we will sometimes denote it  $\Lambda(\mathbf{v})$ . The variety  $\Lambda(V)$  (or  $\Lambda(\mathbf{v})$ ) is called a *Lusztig quiver variety*. It was first defined in [12].

**2.2. Nakajima quiver varieties.** For  $V, W \in \mathcal{V}$  define

$$\mathbf{M}(V, W) = E(V, V) \oplus L(W, V) \oplus L(V, W).$$

The three components of an element of  $\mathbf{M}(V, W)$  will typically be denoted by  $x$ ,  $s$ , and  $t$ . For an  $I$ -graded subspace  $S \subseteq V$  and  $x \in E(V, V)$ , we say that  $S$  is  $x$ -invariant if  $x_h(S_{\text{out}(h)}) \subseteq S_{\text{in}(h)}$  for all  $h \in H$ .

The group  $G_V$  acts on  $\mathbf{M}(V, W)$  by

$$g \cdot (x, s, t) = (gxg^{-1}, gs, tg^{-1}).$$

We have a nondegenerate,  $G_V$ -invariant, symplectic form on  $\mathbf{M}(V, W)$  defined by

$$\omega((x, s, t), (x', s', t')) = \text{tr}((\epsilon x)x') + \text{tr}(st' - s't).$$

The corresponding moment map is given by

$$\mu(x, s, t) = (\epsilon x)x + st.$$

Consider the zero set  $\mu^{-1}(0)$  of  $\mu$ . When we wish to specify  $V$  and  $W$ , we write  $\mu_{V,W}^{-1}(0)$ . This is a (not necessarily irreducible) affine algebraic variety. We say that a point  $(x, s, t) \in \mu^{-1}(0)$  is *stable* if the only  $I$ -graded  $x$ -invariant subspace of  $V$  contained in the kernel of  $t$  is zero. We denote the set of stable points by  $\mu^{-1}(0)^s$ . The action of  $G_V$  on  $\mu^{-1}(0)^s$  is free and the quotient

$$\mathfrak{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)^s / G_V$$

is a nonsingular quasi-projective variety with symplectic form induced by  $\langle \cdot, \cdot \rangle$ . It is labeled by the graded dimensions  $\mathbf{v} = \mathbf{dim} V = (\dim V_i)_{i \in I}$  and  $\mathbf{w} = \mathbf{dim} W = (\dim W_i)_{i \in I}$  of  $V$  and  $W$  since, up to isomorphism, it depends only on these dimensions. A  $G_V$ -orbit through  $(x, s, t)$ , considered as a point of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ , will be denoted  $[x, s, t]$ . We call  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  a *Nakajima quiver variety*. It was originally defined in [16, 17].

Let  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0) // G_V$  be the affine algebro-geometric quotient. That is, it is the affine algebraic variety whose coordinate ring is the  $G_V$ -invariant polynomials on  $\mu^{-1}(0)$ . As a set, it consists of the closed  $G_V$ -orbits in  $\mu^{-1}(0)$ . We have a projective morphism  $\pi : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  which sends  $[x, s, t]$  to the unique closed orbit contained in the closure of the orbit  $G_V \cdot (x, s, t)$ . We then define

$$\mathfrak{L}(\mathbf{v}, \mathbf{w}) = \pi^{-1}(0).$$

It is a lagrangian subvariety of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ . Let

$$\mathfrak{M}(\mathbf{w}) = \bigsqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w}), \quad \mathfrak{L}(\mathbf{w}) = \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w}).$$

**2.3. Tensor product quiver varieties.** Let  $W^i \in \mathcal{V}$ ,  $i = 1, 2, \dots, n$ , with graded dimensions  $\mathbf{w}^i$  and  $V \in \mathcal{V}$  with graded dimension  $\mathbf{v}$ . Set  $W = \bigoplus_{i=1}^n W^i$  and  $W^{j,k} = \bigoplus_{i=j}^k W^i$  for  $1 \leq j \leq k \leq n$ . We adopt the convention that  $W^{j,k} = 0$  if  $j > k$ . The group  $G_W$  acts on  $\mathbf{M}(V, W)$  by

$$g * (x, s, t) = (x, sg^{-1}, gt).$$

This commutes with the action of  $G_V$  and thus induces an action of  $G_W$  on  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  and  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  and the map  $\pi$  is  $G_W$ -equivariant. Define a one-parameter subgroup  $\lambda : \mathbb{C}^* \rightarrow G_W$  by

$$\lambda(z) = \text{id}_{W^1} \oplus z \text{id}_{W^2} \oplus \dots \oplus z^{n-1} \text{id}_{W^n} \in \prod_{i=1}^n GL(W^i) \subseteq G_W.$$

Let  $\mathfrak{M}(\mathbf{v}, \mathbf{w})^{\lambda(\mathbb{C}^*)}$  denote the fixed point set of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ .

**Lemma 2.2** ([19, Lemma 3.2]). *We have*

$$\mathfrak{M}(\mathbf{v}, \mathbf{w})^{\lambda(\mathbb{C}^*)} \cong \bigsqcup_{\sum \mathbf{v}^i = \mathbf{v}} \left( \prod_{i=1}^n \mathfrak{M}(\mathbf{v}^i, \mathbf{w}^i) \right)$$

where the union is over all ordered  $n$ -tuples  $(\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^n)$  such that  $\sum \mathbf{v}^i = \mathbf{v}$ .

Taking the union over all possible  $\mathbf{v}$  yields

$$\mathfrak{M}(\mathbf{w})^{\lambda(\mathbb{C}^*)} \cong \prod_{i=1}^n \mathfrak{M}(\mathbf{w}^i).$$

Define the *tensor product quiver variety*

$$\begin{aligned} \mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n) &= \left\{ [x, s, t] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{z \rightarrow 0} \lambda(z) * [x, s, t] \in \mathfrak{L}(\mathbf{w}^1) \times \dots \times \mathfrak{L}(\mathbf{w}^n) \right\}, \text{ and} \\ \mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n) &= \mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n) \cap \mathfrak{M}(\mathbf{v}, \mathbf{w}). \end{aligned}$$

Note that the limit in the above definition does not always exist. As shown in [19, Lemma 3.6],  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  is a closed subvariety of  $\mathfrak{M}(\mathbf{w})$  and  $\mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  is a closed subvariety of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ . By [19, Lemma 3.5], we also have

$$\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n) = \left\{ [x, s, t] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{z \rightarrow 0} \lambda(z) * \pi([x, s, t]) = 0 \right\}.$$

**Example 2.3** ( $\mathfrak{g} = \mathfrak{sl}_2$ ). *If  $\mathfrak{g} = \mathfrak{sl}_2$ , the corresponding quiver has one vertex and no edges. Therefore  $x = 0$ , the stability condition forces  $t$  to be injective, and the moment map condition implies  $st = 0$ . Then the map*

$$(0, s, t) \mapsto (\text{im } t, ts)$$

*identifies  $\mathfrak{M}(v, w)$  with*

$$\{(S, \chi) \in Gr(v, w) \times \text{End } W \mid \chi(S) = 0, \chi(W) \subseteq S\} \cong T^*Gr(v, w),$$

where  $Gr(v, w)$  is the Grassmannian of dimension  $v$  planes in the  $w$ -dimensional space  $W$ . If we fix a decomposition  $W = \bigoplus_{i=1}^n W^i$ , then  $\mathfrak{T}(w^1, \dots, w^n)$  is the subvariety of  $\mathfrak{M}(v, w)$  consisting of those  $(S, \chi) \in Gr(v, w) \times \text{End } W$  for which  $\chi(W^i) \subseteq W^{i+1} \oplus \dots \oplus W_n$ .



## 3. GEOMETRIC REALIZATIONS OF CRYSTALS

In this section we recall the construction of the crystals  $B_\infty$  and  $B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  on sets of irreducible components of quiver varieties. We also describe a geometric realization of Kashiwara's involution.

**3.1. Geometric realization of  $B_\infty$ .** We briefly review here the geometric realization of the crystal  $B_\infty$  defined by Kashiwara and Saito [10]. We identify  $(\mathbb{Z}_{\geq 0})^I$  with the negative root lattice of  $\mathfrak{g}$  by identifying  $\mathbf{v} = (\mathbf{v}_i)$  with  $-\sum_{i \in I} \mathbf{v}_i \alpha_i$  where  $\alpha_i$  are the simple roots of  $\mathfrak{g}$ . For each  $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^I$ , choose an  $I$ -graded vector space  $V(\mathbf{v})$  of graded dimension  $\mathbf{v}$ . Then let  $\Lambda(\mathbf{v}) = \Lambda(V(\mathbf{v}))$ . Let  $\Lambda(\mathbf{v}', \mathbf{v})$  be the variety of triples  $(x, \phi', \bar{\phi})$  such that  $x \in \Lambda(\mathbf{v})$  and  $\phi' = (\phi'_i)$ ,  $\bar{\phi} = (\bar{\phi}_i)$  give an exact sequence

$$0 \rightarrow V(\mathbf{v}')_i \xrightarrow{\phi'_i} V(\mathbf{v})_i \xrightarrow{\bar{\phi}_i} V(\mathbf{v} - \mathbf{v}')_i \rightarrow 0$$

for each  $i \in I$  and  $\text{im } \phi'$  is  $x$ -invariant. Then  $x$  induces a map  $x' \in \Lambda(\mathbf{v}')$  and so we have the following diagram

$$(3.1) \quad \Lambda(\mathbf{v}') \xleftarrow{q_1} \Lambda(\mathbf{v}', \mathbf{v}) \xrightarrow{q_2} \Lambda(\mathbf{v})$$

where  $q_1(x, \phi', \bar{\phi}) = x'$  and  $q_2(x, \phi', \bar{\phi}) = x$ .

For  $x \in \Lambda(\mathbf{v})$  and  $i \in I$ , let

$$\varepsilon_i(x) = \dim \text{Coker} \left( \bigoplus_{h: \text{in}(h)=i} V(\mathbf{v})_{\text{out}(h)} \xrightarrow{(x_h)} V(\mathbf{v})_i \right)$$

and for  $i \in I$  and  $c \in \mathbb{Z}_{\geq 0}$  let

$$\Lambda(\mathbf{v})_{i,c} = \{x \in \Lambda(\mathbf{v}) \mid \varepsilon_i(x) = c\}.$$

Let  $B(\mathbf{v}, \infty)$  be the set of irreducible components of  $\Lambda(\mathbf{v})$  and for  $X \in B(\mathbf{v}, \infty)$ , define  $\varepsilon_i(X) = \varepsilon_i(x)$  for a generic point  $x$  of  $X$ . For  $i \in I$  and  $c \in \mathbb{Z}_{\geq 0}$ , let

$$B(\mathbf{v}, \infty)_{i,c} = \{X \in B(\mathbf{v}, \infty) \mid \varepsilon_i(X) = c\}.$$

Then (3.1) induces an isomorphism (see [10, Prop 5.2.4])

$$(3.2) \quad B(\mathbf{v} + c\alpha_i, \infty)_{i,0} \cong B(\mathbf{v}, \infty)_{i,c}.$$

Let  $B(\infty) = \bigsqcup_{\mathbf{v}} B(\mathbf{v}, \infty)$ . We define crystal operators on  $B(\infty)$  as follows. Suppose  $\Lambda' \in B(\mathbf{v} + c\alpha_i, \infty)_{i,0}$  corresponds to  $\Lambda \in B(\mathbf{v}, \infty)_{i,c}$  by the isomorphism (3.2). Define

$$\begin{aligned} \tilde{f}_i^c : B(\mathbf{v} + c\alpha_i, \infty)_{i,0} &\rightarrow B(\mathbf{v}, \infty)_{i,c}, & \tilde{f}_i^c(\Lambda') &= \Lambda, \\ \tilde{e}_i^c : B(\mathbf{v}, \infty)_{i,c} &\rightarrow B(\mathbf{v} + c\alpha_i, \infty)_{i,0}, & \tilde{e}_i^c(\Lambda) &= \Lambda'. \end{aligned}$$

For  $c > 0$  we then define  $\tilde{e}_i : B(\infty) \rightarrow B(\infty) \sqcup \{0\}$  by

$$\tilde{e}_i : B(\mathbf{v}, \infty)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} + c\alpha_i, \infty)_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} B(\mathbf{v} + \alpha_i, \infty)_{i,c-1}$$

and let  $\tilde{e}_i(X) = 0$  for  $X \in B(\mathbf{v}, \infty)_{i,0}$ . Also define

$$\tilde{f}_i : B(\infty) \rightarrow B(\infty), \quad \tilde{f}_i : B(\mathbf{v}, \infty)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} + c\alpha_i, \infty)_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(\mathbf{v} - \alpha_i, \infty)_{i,c+1}.$$

Then  $\tilde{e}_i^c$  and  $\tilde{f}_i^c$  can be considered the  $c$ th powers of  $\tilde{e}_i$  and  $\tilde{f}_i$  respectively. If  $P$  is the weight lattice of  $\mathfrak{g}$ , we also define

$$\begin{aligned} \text{wt} : B(\infty) &\rightarrow P; \quad \text{wt}(X) = \mathbf{v} \text{ for } X \in B(\mathbf{v}, \infty), \\ \varphi_i(X) &= \varepsilon_i(X) + \langle h_i, \text{wt}(X) \rangle. \end{aligned}$$

**Proposition 3.1** ([10, Thm 5.2.6, Thm 5.3.2]). *The above definitions endow  $B(\infty)$  with the structure of a  $\mathfrak{g}$ -crystal and  $B(\infty) \cong B_\infty$  as  $\mathfrak{g}$ -crystals.*

For an element  $b \in B_\infty$ , let  $X_b$  denote the corresponding element of  $B(\infty)$ .

**3.2. Geometric realization of Kashiwara's involution.** We recall here a geometric realization, introduced by Kashiwara and Saito [10], of the involution described in Section 1.1. Fix a nondegenerate Hermitian form on  $V(\mathbf{v})_i$  for all  $i$  and  $\mathbf{v}$ . Then  $x \mapsto x^\dagger$ , where  $\dagger$  denotes the Hermitian adjoint, gives an automorphism of  $E(V(\mathbf{v}), V(\mathbf{v}))$  and  $\Lambda(\mathbf{v})$  is invariant under this automorphism. This induces an involution of  $B(\mathbf{v}, \infty)$  which we denote by  $*$ . Since  $\Lambda(\mathbf{v})$  is  $G_{V(\mathbf{v})}$ -invariant, the involution  $*$  does not depend on our choice of Hermitian forms. It was shown in [10] that  $*$  corresponds to Kashiwara's involution under the isomorphism  $B(\infty) \cong B_\infty$ . That is,  $X_b^* = X_{b^*}$  for all  $b \in B_\infty$ . Note that in [10], an isomorphism between  $V(\mathbf{v})_i$  and its dual was chosen for each  $\mathbf{v}$  and  $i$  and the transpose, rather than the Hermitian adjoint, was used to realize Kashiwara's involution. Fixing a real form of each  $V(\mathbf{v})_i$  (i.e. a real vector space  $V(\mathbf{v})_i^{\mathbb{R}}$  such that  $V(\mathbf{v})_i = V(\mathbf{v})_i^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ ), our Hermitian form yields a nondegenerate bilinear form given by  $(u, v) \mapsto \langle u, \kappa(v) \rangle$  where  $\langle \cdot, \cdot \rangle$  is the Hermitian form and  $\kappa : w \otimes z \mapsto w \otimes \bar{z}$ ,  $w \in V(\mathbf{v})_i^{\mathbb{R}}$ ,  $z \in \mathbb{C}$ , is the conjugation determined by the real form. This gives an identification of  $V(\mathbf{v})_i$  with its dual for each  $i$  and Hermitian adjoint corresponds to transpose. In what follows, we will often write  $V$  for  $V(\mathbf{v})$ , when it will cause no confusion, to simplify notation.

**3.3. Geometric realization of  $B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$ .** Malkin [14] and Nakajima [19] have endowed the set of irreducible components of the tensor product quiver variety with the structure of a  $\mathfrak{g}$ -crystal. We briefly recall the construction here.

Let

$$(3.3) \quad \mathfrak{T}(\mathbf{v}', \mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n) = \{(x, s, t, S) \mid [x, s, t] \in \mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n), \\ \text{im } s \subseteq S \subseteq V, S \text{ is } x\text{-invariant}, \mathbf{dim } S = \mathbf{v}'\} / G_V.$$

Then we have the diagram

$$(3.4) \quad \mathfrak{T}(\mathbf{v}'; \mathbf{w}^1, \dots, \mathbf{w}^n) \xleftarrow{q'_1} \mathfrak{T}(\mathbf{v}', \mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n) \xrightarrow{q'_2} \mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$$

where  $q'_1(G_V \cdot (x, s, t, S)) = [x^S, s^{W,S}, t^{S,W}]$  and  $q'_2(G_V \cdot (x, s, t, S)) = [x, s, t]$ . Here  $x^S$  and  $t^{S,W}$  denote the restriction of  $x$  and  $t$  to  $S$  respectively and  $s^{W,S}$  is the map  $s$  viewed as a map into  $S$ .

For  $[x, s, t] \in \mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  and  $i \in I$ , define

$$\varepsilon_i([x, s, t]) = \dim(V_i / \text{im } \tau_i),$$

where

$$\tau_i = \tau_{i, (x, s, t)} = \sum_{h : \text{in}(h) = i} \epsilon(h) x_h + s_i : \bigoplus_{h : \text{in}(h) = i} V_{\text{out}(h)} \oplus W_i \rightarrow V_i.$$

For  $c \in \mathbb{Z}_{\geq 0}$  let

$$\mathfrak{Z}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c} = \{[x, s, t] \in \mathfrak{Z}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n) \mid \varepsilon_i([x, s, t]) = c\}.$$

Then  $\mathfrak{Z}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c}$  is a locally closed subvariety of  $\mathfrak{Z}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$ . Let  $B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  denote the set of irreducible components of  $\mathfrak{Z}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  and let  $B(\mathbf{w}^1, \dots, \mathbf{w}^n) = \bigsqcup_{\mathbf{v}} B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$ . For  $X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  define  $\varepsilon_i(X) = \varepsilon_i([x, s, t])$  for a generic point  $[x, s, t]$  of  $X$ . For  $i \in I$  and  $c \in \mathbb{Z}_{\geq 0}$ , let

$$B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c} = \{X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n) \mid \varepsilon_i(X) = c\}.$$

For  $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^I$ , let  $\lambda_{\mathbf{w}} = \sum_i \mathbf{w}_i \omega_i$  where the  $\omega_i$  are the fundamental weights of  $\mathfrak{g}$ . Then define

$$\begin{aligned} \text{wt} : B(\mathbf{w}^1, \dots, \mathbf{w}^n) &\rightarrow P, \quad \text{wt}(X) = \lambda_{\mathbf{w}} + \mathbf{v} \text{ for } X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n), \\ \varphi_i(X) &= \varepsilon_i(X) + \langle h_i, \text{wt}(X) \rangle. \end{aligned}$$

Note that for  $X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c}$ ,

$$\varphi_i(X) = c + \langle h_i, \lambda_{\mathbf{w}} + \mathbf{v} \rangle = \dim(\ker \tau_i / \text{im } \gamma_i), \quad \text{where } \gamma_i = \bigoplus_{h: \text{in}(h)=i} x_h \oplus t_i.$$

The maps (3.4) induce an isomorphism (see [19, §4])

$$(3.5) \quad B(\mathbf{v} + c\alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0} \cong B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c}.$$

We define crystal operators on  $B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  as follows. Let  $X' \in B(\mathbf{v} + c\alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0}$  correspond to  $X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c}$  under the isomorphism (3.5). Then define

$$\begin{aligned} \tilde{f}_i^c : B(\mathbf{v} + c\alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0} &\rightarrow B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c}, \quad \tilde{f}_i^c(X') = X, \\ \tilde{e}_i^c : B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c} &\rightarrow B(\mathbf{v} + c\alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0}, \quad \tilde{e}_i^c(X) = X'. \end{aligned}$$

For  $c > 0$ , we then define  $\tilde{e}_i : B(\mathbf{w}^1, \dots, \mathbf{w}^n) \rightarrow B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  by

$$\tilde{e}_i : B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} + c\alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} B(\mathbf{v} + \alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c-1},$$

and set  $\tilde{e}_i(X) = 0$  for  $X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0}$ . For  $c > -\langle h_i, \lambda_{\mathbf{w}} + \mathbf{v} \rangle$ , let

$$\tilde{f}_i : B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} + c\alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(\mathbf{v} - \alpha_i; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c+1},$$

and set  $\tilde{f}_i(X) = 0$  for  $X \in B(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)_{i,c}$  with  $c \leq -\langle h_i, \lambda_{\mathbf{w}} + \mathbf{v} \rangle$ . The maps  $\tilde{e}_i^c$  and  $\tilde{f}_i^c$  defined above can be considered the  $c$ th powers of  $\tilde{e}_i$  and  $\tilde{f}_i$  respectively.

**Proposition 3.2** ([19, Prop 4.3, Thm 4.6, Cor 4.7, §7]). *The above definitions endow  $B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  with the structure of a  $\mathfrak{g}$ -crystal and  $B(\mathbf{w}^1, \dots, \mathbf{w}^n) \cong B_{\lambda_{\mathbf{w}^1}} \otimes \dots \otimes B_{\lambda_{\mathbf{w}^n}}$  as  $\mathfrak{g}$ -crystals.*

For  $b \in B_{\lambda_{\mathbf{w}^1}} \otimes \dots \otimes B_{\lambda_{\mathbf{w}^n}}$ , let  $Y_b \in B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  denote the corresponding irreducible component of the tensor product quiver variety  $\mathfrak{Z}(\mathbf{w}^1, \dots, \mathbf{w}^n)$ .

**3.4. Fiber bundles and crystal isomorphisms.** Let  $\tilde{\mathfrak{T}}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  and  $\tilde{\mathfrak{T}}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  denote the inverse images of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  and  $\mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)$  (respectively) under the natural projection  $\mu^{-1}(0)^s \rightarrow \mathfrak{M}(\mathbf{w})$ . That is,

$$\begin{aligned}\tilde{\mathfrak{T}}(\mathbf{w}^1, \dots, \mathbf{w}^n) &= \{(x, s, t) \in \mu^{-1}(0)^s \mid [x, s, t] \in \mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)\}, \\ \tilde{\mathfrak{T}}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n) &= \{(x, s, t) \in \mu^{-1}(0)^s \mid [x, s, t] \in \mathfrak{T}(\mathbf{v}; \mathbf{w}^1, \dots, \mathbf{w}^n)\}.\end{aligned}$$

Since the aforementioned projection is a principle  $G_V$ -bundle, the irreducible components of  $\tilde{\mathfrak{T}}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  are in natural one-to-one correspondence with the irreducible components of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$ . Note that the  $n = 1$  case reduces to  $\mathfrak{T}(\mathbf{w}) = \mathfrak{L}(\mathbf{w})$  and we define  $\tilde{\mathfrak{L}}(\mathbf{w}) = \tilde{\mathfrak{T}}(\mathbf{w})$ . Let  $\tilde{Y}_b$  denote the irreducible component of  $\tilde{\mathfrak{T}}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  corresponding to the irreducible component  $Y_b$  of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$ .

It will be useful to have a slightly more concrete description of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$ . It is shown in [19, Prop 3.8] (while only the case  $n = 2$  is considered there, the generalization to higher  $n$  is straightforward) that  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  decomposes as a disjoint union

$$\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n) = \bigsqcup_{\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^n} \mathfrak{T}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n)$$

where

$$\begin{aligned}\mathfrak{T}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n) \\ \stackrel{\text{def}}{=} \left\{ [x, s, t] \mid \lim_{z \rightarrow 0} \lambda(z) * [x, s, t] \in \mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2) \times \dots \times \mathfrak{L}(\mathbf{v}^n, \mathbf{w}^n) \right\}.\end{aligned}$$

These are the Bialynicki-Birula decompositions of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$ . The map

$$(3.6) \quad \mathfrak{T}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n) \ni [x, s, t] \mapsto \lim_{z \rightarrow 0} \lambda(z) * [x, s, t] \\ \in \mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2) \times \dots \times \mathfrak{L}(\mathbf{v}^n, \mathbf{w}^n)$$

is a fiber bundle with affine fibers. By a generalization of the results of [19, Prop 3.8, Prop 3.15] to more than two factors, these fiber bundles identify the irreducible components of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  with the irreducible components of  $\mathfrak{L}(\mathbf{w}^1) \times \dots \times \mathfrak{L}(\mathbf{w}^n)$  and this identification is an isomorphism of crystals [19, Thm 4.6]. Here we use the tensor product rule and the crystal structure on each  $\mathfrak{L}(\mathbf{w}^i)$  to give a crystal structure to  $\mathfrak{L}(\mathbf{w}^1) \times \dots \times \mathfrak{L}(\mathbf{w}^n)$ . In general, the crystal  $B_{\lambda_{\mathbf{w}^1}} \otimes \dots \otimes B_{\lambda_{\mathbf{w}^n}}$  has nontrivial automorphisms. Therefore, the isomorphism of Proposition 3.2 is not necessarily unique. However, each  $B_{\lambda_{\mathbf{w}^i}}$  has no nontrivial automorphisms since it is generated by a single highest weight element. Therefore we can use the identification of  $B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  with  $B(\mathbf{w}^1) \times \dots \times B(\mathbf{w}^n)$  induced by (3.6) and the unique isomorphisms  $B(\mathbf{w}^i) \cong B_{\lambda_{\mathbf{w}^i}}$  to fix an isomorphism

$$(3.7) \quad \phi : B(\mathbf{w}^1, \dots, \mathbf{w}^n) \xrightarrow{\cong} B_{\lambda_{\mathbf{w}^1}} \otimes \dots \otimes B_{\lambda_{\mathbf{w}^n}}.$$

**Lemma 3.3.** *Suppose  $[x, s, t] \in \mathfrak{T}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n)$ . That is,*

$$\lim_{z \rightarrow 0} \lambda(z) * [x, s, t] \in \mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2) \times \dots \times \mathfrak{L}(\mathbf{v}^n, \mathbf{w}^n).$$

Then there exists a  $d \in \mathbb{Z}_{>0}$ , representatives  $(x^j, s^j, t^j) \in \tilde{\mathfrak{L}}(\mathbf{v}^j, \mathbf{w}^j)$  and a one-parameter subgroup  $\rho : \mathbb{C}^* \rightarrow G_V$  such that

$$(3.8) \quad \lim_{z \rightarrow 0} \lambda(z) * [x, s, t] = ([x^1, s^1, t^1], \dots, [x^n, s^n, t^n])$$

and

$$(3.9) \quad \lim_{z \rightarrow 0} \lambda(z^d) * \rho(z) \cdot (x, s, t) = (x^1 \oplus \dots \oplus x^n, s^1 \oplus \dots \oplus s^n, t^1 \oplus \dots \oplus t^n).$$

*Proof.* Let

$$[x', s', t'] = \lim_{z \rightarrow 0} \lambda(z) * [x, s, t]$$

and fix a representative  $(x', s', t') \in \mu^{-1}(0)^s \subseteq \mathbf{M}(V, W)$ . Then we can write

$$(x', s', t') = (x^1 \oplus \dots \oplus x^n, s^1 \oplus \dots \oplus s^n, t^1 \oplus \dots \oplus t^n)$$

for some  $(x^j, s^j, t^j) \in \tilde{\mathfrak{L}}(\mathbf{v}^j, \mathbf{w}^j)$ ,  $1 \leq j \leq n$ . By the geometric invariant theory (see, for example, [15, Definition 1.7]), there exists an affine  $G_V$ -invariant neighborhood  $U$  of  $(x', s', t')$  in  $\mathbf{M}(V, W)$  such that the orbit  $G_V \cdot (x', s', t')$  is closed in  $U$ . We may assume that  $(x, s, t)$  is in  $U$ . Consider the action of the reductive group  $\mathbb{C}^* \times G_V$  on  $\mathbf{M}(V, W)$  given by

$$((z, g), (x, s, t)) \mapsto (z, g) \star (x, s, t) \stackrel{\text{def}}{=} \lambda(z) * g \cdot (x, s, t).$$

By hypothesis,  $G_V \cdot (x', s', t')$  meets the closure of the orbit  $(\mathbb{C}^* \times G_V) \star (x, s, t)$ . Therefore, by a version of the Hilbert criterion (see [11, Thm 1.4]), since  $U$  is affine, there exists a one-parameter subgroup  $(\rho', \rho) : \mathbb{C}^* \rightarrow \mathbb{C}^* \times G_V$  such that  $\lim_{z \rightarrow 0} (\rho', \rho)(z) \star (x, s, t)$  exists and is contained in  $G_V \cdot (x', s', t')$ . By modification of the one-parameter subgroup (or representative  $(x', s', t')$ ), we may assume that

$$\lim_{z \rightarrow 0} (\rho', \rho)(z) \star (x, s, t) = (x', s', t').$$

We may also assume that  $\rho'(z) \rightarrow 0$  as  $z \rightarrow 0$ . That is  $\rho'(z) = z^d$  for some  $d \in \mathbb{Z}_{>0}$ . Then we have

$$\lim_{z \rightarrow 0} \lambda(z^d) * \rho(z) \cdot (x, s, t) = \lim_{z \rightarrow 0} (\rho', \rho)(z) \star (x, s, t) = (x', s', t')$$

as desired.  $\square$

Note that if we have a flag of  $I$ -graded spaces

$$0 = V^{n+1} \subseteq V^n \subseteq V^{n-1} \subseteq \dots \subseteq V^1 = V,$$

with

$$x(V^i) \subseteq V^i, \quad s(W^i) \subseteq V^i, \quad t(V^i) \subseteq W^{i,n}, \quad 1 \leq i \leq n,$$

then  $x, s$  and  $t$  induce maps

$$x^{V^i/V^{i+1}} : V^i/V^{i+1} \rightarrow V^i/V^{i+1}, \quad s^{W^i, V^i/V^{i+1}} : W^i \rightarrow V^i/V^{i+1}, \quad t^{V^i/V^{i+1}, W^i} : V^i/V^{i+1} \rightarrow W^i$$

**Proposition 3.4.** *The set  $\mathfrak{T}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n)$  consists of those  $[x, s, t]$  in  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  such that there exists a flag of  $I$ -graded spaces*

$$0 = V^{n+1} \subseteq V^n \subseteq V^{n-1} \subseteq \dots \subseteq V^1 = V, \quad \dim V^i/V^{i+1} = \mathbf{v}_i,$$

with

$$x(V^i) \subseteq V^i, \quad s(W^i) \subseteq V^i, \quad t(V^i) \subseteq W^{i,n},$$

and

$$(x^{V^i/V^{i+1}}, s^{W^i, V^i/V^{i+1}}, t^{V^i/V^{i+1}, W^i}) \in \tilde{\mathfrak{L}}(\mathbf{v}^i, \mathbf{w}^i), \quad 1 \leq i \leq n.$$

*Proof.* By Lemma 3.3, there exists a positive integer  $d$ , representatives  $(x^j, s^j, t^j) \in \tilde{\mathfrak{L}}(\mathbf{v}^j, \mathbf{w}^j)$ , and a one-parameter subgroup  $\rho : \mathbb{C}^* \rightarrow G_V$  such that (3.8) and (3.9) hold. Denote the  $\rho$ -weight space decomposition of  $V_i$ ,  $i \in I$ , by

$$V_i = \bigoplus_{m \in \mathbb{Z}} V_i^{(m)}, \quad V_i^{(m)} = \{v \in V_i \mid \rho(z)_i(v) = z^m v, z \in \mathbb{C}^*\}.$$

The sum  $V^{(m)} = \bigoplus_{i \in I} V_i^{(m)}$  is an  $I$ -graded subspace of  $V$ . Then (3.9) implies

$$x(V^{(k)}) \subseteq \bigoplus_{m \geq k} V^{(m)}, \quad s(W^k) \subseteq \bigoplus_{m \geq d(k-1)} V^{(m)}, \quad t(V^{(d(k-1))}) \subseteq W^{k,n}.$$

The stability condition then implies that  $V^{(l)} = 0$  for  $l \geq dn$  and the flag given by

$$V^k = \bigoplus_{m \geq d(k-1)} V^{(m)}, \quad 1 \leq k \leq n.$$

satisfies the conditions of the proposition.

Conversely, suppose that for some  $[x, s, t] \in \mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$ , a flag with the given properties exists. For each  $i \in I$ , choose a decomposition

$$(3.10) \quad V_i = \bigoplus_{m=1}^n V_i^{(m)}$$

such that

$$V_i^k = \bigoplus_{m \geq k} V_i^{(m)}, \quad 1 \leq k \leq n.$$

Then define a one-parameter subgroup  $\rho : \mathbb{C}^* \rightarrow G_V$  by

$$\rho(z)_i = \sum_{m=1}^n z^{m-1} \text{id}_{V_i^{(m)}}.$$

Then it is easily seen that

$$\lim_{z \rightarrow 0} \lambda(z) * \rho(z) \cdot (x, s, t) = (x^{V^{(1)}} \oplus \dots \oplus x^{V^{(n)}}, s^{W^1, V^{(1)}} \oplus \dots \oplus s^{W^n, V^{(n)}}, t^{V^{(1)}, W^1} \oplus \dots \oplus t^{V^{(n)}, W^n})$$

where  $x^{V^{(k)}}$  denotes the restriction of  $x$  to  $V^{(k)}$  composed with the projection to  $V^{(k)}$  (according to the decomposition given in (3.10)). The maps  $s^{W^k, V^k}$  and  $t^{V^k, W^k}$  are defined similarly. Thus

$$\lim_{z \rightarrow 0} \lambda(z) * [x, s, t] = ([x^1, s^1, t^1], \dots, [x^n, s^n, t^n]).$$

where

$$[x^k, s^k, t^k] = [x^{V^k/V^{k+1}}, s^{W^k, V^k/V^{k+1}}, t^{V^k/V^{k+1}, W^k}] \in \mathfrak{L}(\mathbf{v}^k, \mathbf{w}^k).$$

□

We define

$$\tilde{\mathfrak{T}}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n) = \{(x, s, t) \mid [x, s, t] \in \mathfrak{T}(\mathbf{v}^1, \dots, \mathbf{v}^n | \mathbf{w}^1, \dots, \mathbf{w}^n)\}.$$

For  $0 = p_0 < p_1 \leq p_2 \leq \dots \leq p_k \leq n$ , define a more general one-parameter subgroup  $\lambda^{(p_1, \dots, p_k)} : \mathbb{C}^* \rightarrow G_W$  by

$$\lambda^{(p_1, \dots, p_k)}(z) = \text{id}_{W^{1, p_1}} \oplus z \text{id}_{W^{p_1+1, p_2}} \oplus \dots \oplus z^k \text{id}_{W^{p_k+1, n}}.$$

Then, as above,  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  decomposes as a disjoint union

$$\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n) = \bigsqcup_{\mathbf{v}^1, \dots, \mathbf{v}^{k+1}} \mathfrak{T}^{(p_1, \dots, p_k)}(\mathbf{v}^1, \dots, \mathbf{v}^{k+1} | \mathbf{w}^1, \dots, \mathbf{w}^n)$$

where

$$(3.11) \quad \mathfrak{T}^{(p_1, \dots, p_k)}(\mathbf{v}^1, \dots, \mathbf{v}^{k+1} | \mathbf{w}^1, \dots, \mathbf{w}^n) \stackrel{\text{def}}{=} \left\{ [x, s, t] \mid \lim_{z \rightarrow 0} \lambda^{(p_1, \dots, p_k)}(z) * [x, s, t] \in \mathfrak{T}(\mathbf{v}^1; \mathbf{w}^1, \dots, \mathbf{w}^{p_1}) \times \dots \times \mathfrak{T}(\mathbf{v}^{k+1}; \mathbf{w}^{p_k+1}, \dots, \mathbf{w}^n) \right\}.$$

The map

$$(3.12) \quad \mathfrak{T}^{(p_1, \dots, p_k)}(\mathbf{v}^1, \dots, \mathbf{v}^{k+1} | \mathbf{w}^1, \dots, \mathbf{w}^n) \ni [x, s, t] \mapsto \lim_{z \rightarrow 0} \lambda^{(p_1, \dots, p_k)}(z) * [x, s, t] \in \mathfrak{T}(\mathbf{v}^1; \mathbf{w}^1, \dots, \mathbf{w}^{p_1}) \times \dots \times \mathfrak{T}(\mathbf{v}^{k+1}; \mathbf{w}^{p_k+1}, \dots, \mathbf{w}^n)$$

is a fiber bundle with affine fibers. These fiber bundles identify the irreducible components of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  with the irreducible components of  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^{p_1}) \times \dots \times \mathfrak{T}(\mathbf{w}^{p_k+1}, \dots, \mathbf{w}^n)$  and this identification is an isomorphism of crystals. We then have the following generalization of Proposition 3.4.

**Proposition 3.5.** *The set  $\mathfrak{T}^{(p_1, \dots, p_k)}(\mathbf{v}^1, \dots, \mathbf{v}^{k+1} | \mathbf{w}^1, \dots, \mathbf{w}^n)$  consists of those  $[x, s, t]$  in  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  such that there exists a flag of  $I$ -graded spaces*

$$0 = V^{k+2} \subseteq V^{k+1} \subseteq V^k \subseteq \dots \subseteq V^1 = V, \quad \dim V^i / V^{i+1} = \mathbf{v}^i,$$

with

$$x(V^i) \subseteq V^i, \quad s(W^{p_{i-1}+1, p_i}) \subseteq V^i, \quad t(V^i) \subseteq W^{p_{i-1}+1, n},$$

and

$$(x^{V^i/V^{i+1}}, s^{W^{p_{i-1}+1, p_i}, V^i/V^{i+1}}, t^{V^i/V^{i+1}, W^{p_{i-1}+1, p_i}}) \in \tilde{\mathfrak{T}}(\mathbf{v}^i; \mathbf{w}^{p_{i-1}+1}, \dots, \mathbf{w}^{p_i}), \quad 1 \leq i \leq n.$$

*Proof.* This follows from Proposition 3.4. The details are left to the reader.  $\square$

**Proposition 3.6.** *If  $(x, s, t) \in \tilde{\mathfrak{T}}(\mathbf{w}^1; \dots; \mathbf{w}^n)$  (equivalently,  $[x, s, t] \in \mathfrak{T}(\mathbf{w}^1; \dots; \mathbf{w}^n)$ ) then  $x$  is nilpotent.*

*Proof.* This follows from Proposition 3.4 and the fact that  $x$  is nilpotent for all  $(x, s, t) \in \tilde{\mathfrak{L}}(\mathbf{v}^i, \mathbf{w}^i)$  (see the proof of [16, Lemma 5.9]).  $\square$

#### 4. A GEOMETRIC COMMUTOR

In this section, we give precise characterizations of the irreducible components of the tensor product quiver variety. We then examine the action of the crystal commutor in terms of these characterizations. This will enable us to prove that the commutor satisfies the cactus relation in Section 5.

**4.1. Characterization of irreducible components.** For an arbitrary element  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$ , let  $\text{hw } b$  be the unique highest weight element in the connected component of the crystal graph of  $B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  containing  $b$ . If  $b'$  is a highest weight element in  $B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$ , then we can identify the connected component containing  $b'$  with some  $B_\lambda$  and  $b'$  corresponds to  $b_\lambda$  under this identification. For  $b \in B_\infty$  and  $b'$  as above, we then define

$$\tilde{b}b' = \begin{cases} b'' & \text{if } \iota_\lambda^\infty(b'') = b, \\ 0 & \text{if } \nexists b'' \text{ such that } \iota_\lambda^\infty(b'') = b, \end{cases}$$

(this is well defined since  $\iota_\lambda^\infty$  is injective). Equivalently,  $\tilde{b}b' = \Phi_\lambda(b)$ , where  $\Phi_\lambda : B_\infty \rightarrow B_\lambda \sqcup \{0\}$  is the natural crystal morphism sending  $b_\infty$  to  $b_\lambda$ . We view  $\tilde{b}b'$  as an element of  $B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  via the above identification. Note that for  $b \in B_\lambda$ ,  $b_1 = \iota^\infty(b)$ , we have  $b = \tilde{b}_1 b_\lambda$ .

**Definition 4.1.** (1) For  $k \geq 1$  and  $0 = p_0 < p_1 \leq p_2 \leq \cdots \leq p_k \leq p_{k+1} = n$  and  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  we define

$$b_{(p_1, \dots, p_k)} = (b_1, b_{\nu_1}, b_2, b_{\nu_2}, \dots, b_{k+1}, b_{\nu_{k+1}})$$

where, for  $1 \leq i \leq k+1$ ,  $b_i \in B_\infty$  and  $b_{\nu_i}$  is a highest weight element in  $B_{\lambda_{p_{i-1}+1}} \otimes \cdots \otimes B_{\lambda_{p_i}}$  of weight  $\nu_i$  as follows. We define the  $b_i$  and  $b_{\nu_i}$  (as well as intermediate elements  $a_i$  and  $b'_i$ ) recursively. First set

$$a_1 = \text{hw } b, \quad b_1 = \iota^\infty(b).$$

Now assume that we have defined a highest weight element  $a_i$  of  $B_{\lambda_{p_{i-1}+1}} \otimes \cdots \otimes B_{\lambda_n}$ . If  $i = k+1$ , we set  $b_{\nu_{k+1}} = a_{k+1}$ . Otherwise we have

$$a_i = b_{\nu_i} \otimes b'_i$$

for a highest weight element  $b_{\nu_i} \in B_{\lambda_{p_{i-1}+1}} \otimes \cdots \otimes B_{\lambda_{p_i}}$  of weight  $\nu_i$  and  $b'_i \in B_{\lambda_{p_i+1}} \otimes \cdots \otimes B_{\lambda_n}$  with  $\varepsilon(b'_i) \leq \nu_i$ . We then define

$$a_{i+1} = \text{hw } b'_i, \quad b_{i+1} = \iota^\infty(b'_i).$$

Thus we have

$$b = \tilde{b}_1(b_{\nu_1} \otimes \tilde{b}_2(b_{\nu_2} \otimes \tilde{b}_3(\cdots \otimes \tilde{b}_{k+1}b_{\nu_{k+1}})\cdots)).$$

(2) For  $k \geq 1$  and  $0 = p_0 < p_1 \leq p_2 \leq \cdots \leq p_k \leq p_{k+1} = n$  and  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  we define

$$b^{(p_1, \dots, p_k)} = (b^1, b^{\nu^1}, b^2, b^{\nu^2}, \dots, b^{k+1}, b^{\nu^{k+1}})$$

where, for  $1 \leq i \leq k+1$ ,  $b^i$  are the unique elements of  $B_\infty$  and  $b^{\nu^i}$  are the unique highest weight elements of  $B_{\lambda_{p_{i-1}+1}} \otimes \cdots \otimes B_{\lambda_{p_i}}$  of weight  $\nu^i$  such that

$$b = \tilde{b}^1(b^{\nu^1} \otimes \tilde{b}^2 b^{\nu^2} \otimes \cdots \otimes \tilde{b}^k b^{\nu^k} \otimes \tilde{b}^{k+1} b^{\nu^{k+1}}).$$

Note that in the case  $k = 1$ , we have  $b^{(p)} = b_{(p)}$  for  $1 \leq p \leq n$ . Also, if  $p_i = p_{i+1}$  for some  $i$ , then we have a trivial tensor product crystal appearing in the above definitions and we set  $b_{i+1} = b^{i+1} = b_\infty$ ,  $\nu_{i+1} = \nu^{i+1} = 0$  and  $b_{\nu_{i+1}} = b^{\nu^{i+1}} = 0$ . In particular,  $b^{(n)} = b_{(n)}$  is always of the form  $(b_1, b_{\nu_1}, b_\infty, 0)$  where  $b = \tilde{b}_1 b_{\nu_1}$ .



Whenever we refer to a sequence  $(p_1, \dots, p_k)$  as above, we will always adopt the convention that  $p_0 = 0$  and  $p_{k+1} = n$ . If for some  $V \in \mathcal{V}$  we have a flag

$$0 = V^{k+2} \subseteq V^{\nu^{k+1}} \subseteq V^{k+1} \subseteq V^{\nu^k} \subseteq V^k \subseteq \dots \subseteq V^{\nu^2} \subseteq V^2 \subseteq V^{\nu^1} \subseteq V^1 = V$$

of  $I$ -graded subspaces and  $0 = p_0 < p_1 \leq \dots \leq p_k \leq p_{k+1} = n$ , we say  $(x, s, t) \in \mathbf{M}(V, W)$   $(p_1, \dots, p_k)$ -*respects* the flag if for all  $1 \leq i \leq k+1$  we have

$$x(V^i) \subseteq V^i, \quad x(V^{\nu^i}) \subseteq V^{\nu^i}, \quad s(W^{p_{i-1}+1, p_i}) \subseteq V^{\nu^i}, \quad t(V^i) \subseteq W^{p_{i-1}+1, n}.$$

We say the flag is  $(p_1, \dots, p_k)$ -*respected* by  $(x, s, t)$ . In this case,  $(x, s, t)$  induces maps

$$\begin{aligned} x^{V^{\nu^i}/V^{i+1}} : V^{\nu^i}/V^{i+1} &\rightarrow V^{\nu^i}/V^{i+1}, \\ x^{V^i/V^{\nu^i}} : V^i/V^{\nu^i} &\rightarrow V^i/V^{\nu^i} \\ s^{W^{p_{i-1}+1, p_i}, V^{\nu^i}/V^{i+1}} : W^{p_{i-1}+1, p_i} &\rightarrow V^{\nu^i}/V^{i+1}, \text{ and} \\ t^{V^{\nu^i}/V^{i+1}, W^{p_{i-1}+1, p_i}} : V^{\nu^i}/V^{i+1} &\rightarrow W^{p_{i-1}+1, p_i}. \end{aligned}$$

For  $b \in B_{\lambda_1} \otimes \dots \otimes B_{\lambda_n}$ , let

$$\tilde{T}_b = \{(x, s, t) \in \tilde{Y}_b \mid \varepsilon_i(x, s, t) = \varepsilon_i(Y_b) = \varepsilon_i(b) \ \forall i \in I\}.$$

Then  $\tilde{T}_b$  is a dense subset of  $\tilde{Y}_b$ .

For  $b_i \in B_\infty$  and highest weight elements  $b_{\nu_i} \in B_{\nu_i}$ ,  $1 \leq i \leq k+1$ , consider the diagram

$$(4.1) \quad \Lambda(\mathbf{v}_1) \times \tilde{\mathfrak{Z}}(\mathbf{v}_{\nu_1}; \mathbf{w}^1, \dots, \mathbf{w}^{p_1}) \times \dots \times \Lambda(\mathbf{v}_{k+1}) \times \tilde{\mathfrak{Z}}(\mathbf{v}_{\nu_{k+1}}; \mathbf{w}^{p_{k+1}}, \dots, \mathbf{w}^n) \\ \xleftarrow{\pi_1} \tilde{\mathfrak{F}}^{(p_1, \dots, p_n)}(\mathbf{w}^1, \dots, \mathbf{w}^n) \xrightarrow{\pi_2} \tilde{\mathfrak{Z}}(\mathbf{w}^1, \dots, \mathbf{w}^n),$$

where, for  $1 \leq i \leq k+1$ ,  $\mathbf{v}_i$  is the weight of  $b_i$ ,  $\nu_i = \sum_{j=p_{i-1}+1}^{p_i} \lambda_{\mathbf{w}^j} + \mathbf{v}_{\nu_i}$  is the weight of  $b_{\nu_i}$ , and  $\tilde{\mathfrak{F}}^{(p_1, \dots, p_n)}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  is the variety parameterizing pairs of  $(x, s, t) \in \tilde{\mathfrak{Z}}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  and flags

$$0 = V^{k+2} \subseteq V^{\nu_{k+1}} \subseteq V^{k+1} \subseteq \dots \subseteq V^{\nu_2} \subseteq V^2 \subseteq V^{\nu_1} \subseteq V^1 = V$$

$(p_1, \dots, p_k)$ -respected by  $(x, s, t)$  with dimensions prescribed by

$$\dim V^i/V^{\nu_i} = \mathbf{v}_i, \quad \dim V^{\nu_i}/V^{i+1} = \mathbf{v}_{\nu_i}, \quad 1 \leq i \leq k+1.$$

The projection  $\pi_2$  forgets the flag, while  $\pi_1$  is given by assigning the corresponding induced maps to  $(x, s, t)$  and the flag as above.

**Definition 4.2.** Let  $0 = p_0 < p_1 \leq \dots \leq p_k \leq p_{k+1} = n$ . For  $b_i \in B_\infty$  and highest weight elements  $b_{\nu_i} \in B_{\nu_i}$ ,  $1 \leq i \leq k+1$ , let  $\pi_1$  and  $\pi_2$  be the projections of (4.1). Define  $\mathcal{Y}^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$  to be the set of irreducible components contained in the closure of

$$\pi_2(\pi_1^{-1}(X_{b_1} \times \tilde{T}_{b_{\nu_1}} \times \dots \times X_{b_{k+1}} \times \tilde{T}_{b_{\nu_{k+1}}}))$$

Note that, a priori,  $\mathcal{Y}^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$  may be empty or consist of several irreducible components.

**Lemma 4.3.** For  $b \in B_{\lambda_1} \otimes \dots \otimes B_{\lambda_n}$  with  $b^{(n)} = b_{(n)} = (b_1, b_{\nu_1}, b_\infty, 0)$ , the set  $\mathcal{Y}^{(n)}(b_1, b_{\nu_1}, b_\infty, 0)$  consists of the single irreducible component  $Y_b$ .

*Proof.* Note that we will always take  $V^{\nu_2} = V^2 = 0$  and so it suffices to consider the subspace  $V^{\nu_1} \subseteq V^1 = V$ . The condition in Definition 4.2 becomes that  $x(V^{\nu_1}) \subseteq V^{\nu_1}$ ,  $s(W) \subseteq V^{\nu_1}$  and

$$(x^{V^{\nu_1}}, s^{W, V^{\nu_1}}, t^{V^{\nu_1}, W}) \in \tilde{T}_{b_{\nu_1}}, \quad x^{V/V^{\nu_1}} \in X_{b_1}.$$

Now, for  $(x', s', t') \in \tilde{T}_{b_{\nu_1}}$ , we have  $\varepsilon_i(x', s', t') = 0$  for all  $i \in I$ . Thus for all  $(x, s, t)$  and  $V^{\nu_1}$  as above, the smallest  $x$ -invariant  $I$ -graded subspace of  $V$  containing  $\text{im } s$  is  $V^{\nu_1}$  (see Proposition 3.6 and Lemma 5.1). Therefore,  $x$ -invariant subspaces of  $V$  containing  $\text{im } s$  are in natural one-to-one correspondence with  $x^{V/V^{\nu_1}}$ -invariant subspaces of  $V/V^{\nu_1}$ . We now show that

$$\mathcal{Y}^{(n)}(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_{\infty}, b_{\nu_1}, b_{\infty}, 0) = \{Y_{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_{\nu_1}}\}$$

(provided  $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_{\nu_1} \neq 0$ ) by induction on  $l$ . In the case  $l = 0$ , we take  $V^{\nu_1} = V$  and the statement holds trivially. Now assume that the result holds for some  $l$ . For all  $i \in I$ , since  $\iota_{\lambda}^{\infty}$  is  $\tilde{e}_i$ -equivariant, we have

$$\varepsilon_i(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} Y_{b_{\nu_1}}) = \varepsilon_i(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_{\nu_1}) = \varepsilon_i(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_{\infty}) = \varepsilon_i(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} X_{b_{\infty}}).$$

Then, upon comparison of the definition of the crystal operators on  $B(\infty)$  and  $B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  (see Sections 3.1 and 3.3), we see that

$$\mathcal{Y}^{(n)}(\tilde{f}_{i_{l+1}} \cdots \tilde{f}_{i_1} b_{\infty}, b_{\nu_1}, b_{\infty}, 0) = \{Y_{\tilde{f}_{i_{l+1}} \cdots \tilde{f}_{i_1} b_{\nu_1}}\}$$

(provided  $\tilde{f}_{i_{l+1}} \cdots \tilde{f}_{i_1} b_{\nu_1} \neq 0$ ). □

**Proposition 4.4.** *Let  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  with  $b^{(p_1, \dots, p_k)} = (b^1, b^{\nu_1}, \dots, b^{k+1}, b^{\nu_{k+1}})$  and  $b_{(p_1, \dots, p_k)} = (b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$ . Then*

$$\mathcal{Y}^{(p_1, \dots, p_k)}(b^1, b^{\nu_1}, \dots, b^{k+1}, b^{\nu_{k+1}}) \text{ and } \mathcal{Y}^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$$

*each consist of the single irreducible component  $Y_b$ .*

*Proof.* It follows from Definition 4.1 that  $b_1 = b^1$ . We first prove the case  $b^1 = b_{\infty}$ , i.e.  $b$  is highest weight. Consider  $\mathcal{Y}^{(p_1, \dots, p_k)}(b^1, b^{\nu_1}, \dots, b^{k+1}, b^{\nu_{k+1}})$ . Now,

$$b = b^{\nu_1} \otimes \tilde{b}^2 b^{\nu_2} \otimes \cdots \otimes \tilde{b}^k b^{\nu_k} \otimes \tilde{b}^{k+1} b^{\nu_{k+1}}.$$

Therefore, by Proposition 3.5 and the fact that the fiber bundle (3.12) induces a crystal isomorphism, we see that  $Y_b$  is the unique irreducible component such that for all  $[x, s, t]$  in a dense subset there is a flag of  $I$ -graded spaces

$$0 = V^{k+2} \subseteq V^{k+1} \subseteq \cdots \subseteq V^2 \subseteq V^{\nu_1} = V$$

with

$$x(V^i) \subseteq V^i, \quad s(W^{p_{i-1}+1, p_i}) \subseteq V^i, \quad t(V^i) \subseteq W^{p_{i-1}+1, n}, \quad 2 \leq i \leq k+1,$$

and

$$(x^{V^i/V^{i+1}}, s^{W^{p_{i-1}+1, p_i}, V^i/V^{i+1}}, t^{V^i/V^{i+1}, W^{p_{i-1}+1, p_i}}) \in \tilde{Y}_{\tilde{b}^i b^{\nu_i}}, \quad 2 \leq i \leq k+1,$$

$$(x^{V^{\nu_1}/V^2}, s^{W^{1, p_1}, V^{\nu_1}/V^2}, t^{V^{\nu_1}/V^2, W^{1, p_1}}) \in \tilde{Y}_{b^{\nu_1}}.$$

Then, applying Lemma 4.3 to describe each  $\tilde{Y}_{\tilde{b}^i b^{\nu^i}}$  for  $2 \leq i \leq k+1$ , we have that  $Y_b$  is the unique irreducible component such that in a dense subset there exists a flag as above and an  $I$ -graded subspace  $\bar{V}^{\nu^i} \subseteq V^i/V^{i+1}$  for  $2 \leq i \leq k+1$  such that

$$x^{V^i/V^{i+1}}(\bar{V}^{\nu^i}) \subseteq \bar{V}^{\nu^i}, \quad s^{W^{p_{i-1}+1, p_i}, V^i/V^{i+1}}(W^{p_{i-1}+1, p_i}) \subseteq \bar{V}^{\nu^i}$$

and if  $V^{\nu^i}$  is the preimage of  $\bar{V}^{\nu^i}$  under the quotient map  $V^i \rightarrow V^i/V^{i+1}$  then

$$(x^{V^{\nu^i}/V^{i+1}}, s^{W^{p_{i-1}+1, p_i}, V^{\nu^i}/V^{i+1}}, t^{V^{\nu^i}/V^{i+1}, W^{p_{i-1}+1, p_i}}) \in \tilde{T}_{b^{\nu^i}}, \quad x^{V^i/V^{\nu^i}} \in X_{b^i}.$$

Considering the flag

$$0 = V^{k+1} \subseteq V^{\nu^k} \subseteq V^k \subseteq \cdots \subseteq V^{\nu^1},$$

we have the result for  $b^1 = b_\infty$ .

We now consider  $\mathcal{Y}^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$ ,  $b_1 = b_\infty$ . We have that

$$b = b_{\nu_1} \otimes \tilde{b}_2(b_{\nu_2} \otimes \tilde{b}_3(\cdots \otimes \tilde{b}_{k+1}b_{\nu_{k+1}}) \cdots).$$

We prove the result by induction on  $k$ . The case  $k = 0$  is just Lemma 4.3. For  $k \geq 1$ , by Proposition 3.5 and the fact that the fiber bundle (3.12) induces a crystal isomorphism, we see that  $Y_b$  is the unique irreducible component such that for all  $[x, s, t]$  in a dense subset there is an  $I$ -graded subspace  $V^2 \subseteq V^{\nu_1}$  such that  $x(V^2) \subseteq V^2$ ,  $s(W^{p_1+1, n}) \subseteq V^2$ ,  $t(V^2) \subseteq W^{p_1+1, n}$ , and

$$(x^{V^{\nu_1}/V^2}, s^{W^{1, p_1}, V^{\nu_1}/V^2}, t^{V^{\nu_1}/V^2, W^{1, p_1}}) \in \tilde{Y}_{b_{\nu_1}}, \quad (x^{V^2}, s^{W^{p_1+1, n}, V^2}, t^{V^2, W^{p_1+1, n}}) \in \tilde{Y}_{b'}$$

where  $b' = \tilde{b}_2(b_{\nu_2} \otimes \tilde{b}_3(\cdots \otimes \tilde{b}_{k+1}b_{\nu_{k+1}}) \cdots)$ . The result then follows by the induction hypothesis.

We now prove the general case  $b^1 = \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_\infty$ , that is

$$\mathcal{Y}^{(p_1, \dots, p_k)}(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_\infty, b^{\nu^1}, \dots, b^{k+1}, b^{\nu^{k+1}}) = \{Y_{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \text{ hw } b}\},$$

$$\mathcal{Y}^{(p_1, \dots, p_k)}(\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} b_\infty, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}}) = \{Y_{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \text{ hw } b}\},$$

for all  $l$  (provided  $b = \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \text{ hw } b \neq 0$ ). The case  $l = 0$  is what we have just proved. The inductive step is analogous to the one in the proof of Lemma 4.3 and is therefore omitted.  $\square$

We denote the unique element of  $\mathcal{Y}^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$  by  $Y^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$ . Note that this is only defined if there exists a  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  with  $b^{(p_1, \dots, p_k)}$  or  $b_{(p_1, \dots, p_k)}$  equal to  $(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$ . From now on, when we write  $Y^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$  we will presuppose the existence of such a  $b$ .

**Corollary 4.5.** *If  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  with  $b^{(p_1, \dots, p_k)} = (b^1, b^{\nu^1}, \dots, b^{k+1}, b^{\nu^{k+1}})$  and  $b_{(p_1, \dots, p_k)} = (b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}})$  then*

$$Y^{(p_1, \dots, p_k)}(b^1, b^{\nu^1}, \dots, b^{k+1}, b^{\nu^{k+1}}) = Y^{(p_1, \dots, p_k)}(b_1, b_{\nu_1}, \dots, b_{k+1}, b_{\nu_{k+1}}).$$

*Proof.* This follows immediately from Proposition 4.4.  $\square$

We note the difference between the two descriptions of  $Y_b$  in Proposition 4.4. Recall the fact, which follows easily from the tensor product rule for crystals, that any highest weight element  $b \in B_\lambda \otimes B_\mu$  is of the form  $b_\lambda \otimes b'$  where  $b_\lambda$  is the highest weight element of  $B_\lambda$  and

$b' \in B_\mu$  with  $\varepsilon(b') \leq \lambda$ . The first description in Proposition 4.4 gives the component  $Y_b$  in terms of the expression of  $b$  in the form

$$b = \tilde{b}_1(b_{\nu_1} \otimes \tilde{b}_2(b_{\nu_2} \otimes \tilde{b}_3(\cdots \otimes \tilde{b}_{k+1}b_{\nu_{k+1}}) \cdots))$$

whereas the second describes the same irreducible component in terms of the expression of  $b$  in the form

$$b = \tilde{b}^1(b^{\nu^1} \otimes \tilde{b}^2b^{\nu^2} \otimes \cdots \otimes \tilde{b}^kb^{\nu^k} \otimes \tilde{b}^{k+1}b^{\nu^{k+1}}).$$

These two expressions are obtained from repeatedly applying the above fact to the different bracketings of the tensor product

$$(B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_{p_1}}) \otimes ((B_{\lambda_{p_1+1}} \otimes \cdots \otimes B_{\lambda_{p_2}}) \otimes \cdots \otimes (B_{\lambda_{p_k+1}} \otimes \cdots \otimes B_{\lambda_n}) \cdots), \quad \text{and} \\ (\cdots (B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_{p_1}}) \otimes \cdots \otimes (B_{\lambda_{p_{k-1}+1}} \otimes \cdots \otimes B_{\lambda_{p_k}})) \otimes (B_{\lambda_{p_k+1}} \otimes \cdots \otimes B_{\lambda_n})$$

respectively.

**4.2. Action of the commutator on tensor product quiver varieties.** We use the isomorphism  $\phi$  of (3.7) to define the action of the crystal commutator on  $B(\mathbf{w}^1, \dots, \mathbf{w}^n)$ . In particular, for  $1 \leq p \leq q < r \leq n$  we define

$$\sigma_{p,q,r} : B(\mathbf{w}^1, \dots, \mathbf{w}^n) \rightarrow B(\mathbf{w}^1, \dots, \mathbf{w}^{p-1}, \mathbf{w}^{q+1}, \dots, \mathbf{w}^r, \mathbf{w}^p, \dots, \mathbf{w}^q, \mathbf{w}^{r+1}, \dots, \mathbf{w}^n), \\ \sigma_{p,q,r} = \phi^{-1} \circ (\text{id}^{\otimes(p-1)} \otimes \sigma_{B_{\lambda_p} \otimes \cdots \otimes B_{\lambda_q}, B_{\lambda_{q+1}} \otimes \cdots \otimes B_{\lambda_r}} \otimes \text{id}^{\otimes(n-r)}) \circ \phi,$$

where  $\lambda_i = \lambda_{\mathbf{w}^i}$  for  $1 \leq i \leq n$ . When  $n = 2$ , we write  $\sigma : B(\mathbf{w}^1, \mathbf{w}^2) \rightarrow B(\mathbf{w}^2, \mathbf{w}^1)$  for  $\sigma_{1,1,2}$ .

**Proposition 4.6.** *Let  $b \in B_{\lambda_{\mathbf{w}^1}} \otimes B_{\lambda_{\mathbf{w}^2}}$  with  $b^{(p)} = b_{(p)} = (b_1, b_{\nu_1}, b_2, b_{\nu_2})$ . Then  $\sigma(\mathcal{Y}^{(p)}(b_1, b_{\nu_1}, b_2, b_{\nu_2}))$  consists of a single element and coincides with  $\mathcal{Y}^{(p)}(b_1, b_{\nu_2}, b_2^*, b_{\nu_1})$ .*

*Proof.* We have

$$\begin{aligned} \sigma(Y^{(p)}(b_1, b_{\nu_1}, b_2, b_{\nu_2})) &= \phi^{-1} \sigma_{B_{\lambda_{\mathbf{w}^1}} \otimes B_{\lambda_{\mathbf{w}^2}}} \phi(Y^{(p)}(b_1, b_{\nu_1}, b_2, b_{\nu_2})) \\ &= \phi^{-1} \sigma_{B_{\lambda_{\mathbf{w}^1}} \otimes B_{\lambda_{\mathbf{w}^2}}} (\tilde{b}_1(b_{\nu_1} \otimes \tilde{b}_2 b_{\nu_2})) \\ &= \phi^{-1} \tilde{b}_1 \sigma_{B_{\lambda_{\mathbf{w}^1}} \otimes B_{\lambda_{\mathbf{w}^2}}} (b_{\nu_1} \otimes \tilde{b}_2 b_{\nu_2}) \\ &= \tilde{b}_1 \phi^{-1} (b_{\nu_2} \otimes \tilde{b}_2^* b_{\nu_1}) \\ &= \phi^{-1} (\tilde{b}_1(b_{\nu_2} \otimes \tilde{b}_2^* b_{\nu_1})) \\ &= Y^{(p)}(b_1, b_{\nu_2}, b_2^*, b_{\nu_1}), \end{aligned}$$

and the result follows.  $\square$

**Proposition 4.7.** *Let  $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$  with  $b_{(p_1, p_2)} = (b_1, b_{\nu_1}, b_2, b_{\nu_2}, b_3, b_{\nu_3})$  and  $b^{(p_1, p_2)} = (b^1, b^{\nu^1}, b^2, b^{\nu^2}, b^3, b^{\nu^3})$ . Then*

$$\sigma_{1, p_1, n} \circ \sigma_{p_1+1, p_2, n}(\mathcal{Y}^{(p_1, p_2)}(b_1, b_{\nu_1}, b_2, b_{\nu_2}, b_3, b_{\nu_3})) \quad \text{and} \\ \sigma_{1, p_2, n} \circ \sigma_{1, p_1, p_2}(\mathcal{Y}^{(p_1, p_2)}(b^1, b^{\nu^1}, b^2, b^{\nu^2}, b^3, b^{\nu^3}))$$

*each consist of a single element and coincide with*

$$\mathcal{Y}^{(p_1, p_2)}(b_1, b_{\nu_3}, b_3^*, b_{\nu_2}, b_2^*, b_{\nu_1}) \quad \text{and} \quad \mathcal{Y}^{(p_1, p_2)}(b^1, b^{\nu^3}, (b^3)^*, b^{\nu^2}, (b^2)^*, b^{\nu^1})$$

*respectively.*

*Proof.* We have

$$\begin{aligned}
& \sigma_{1,p_1,n} \circ \sigma_{p_1+1,p_2,n}(Y^{(p_1,p_2)}(b_1, b_{\nu_1}, b_2, b_{\nu_2}, b_3, b_{\nu_3})) \\
&= \phi^{-1}(\sigma_{B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n} \otimes B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}}) \\
&\quad (\text{id}^{\otimes p_1} \otimes \sigma_{B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n}}) \phi(Y^{(p_1,p_2)}(b_1, b_{\nu_1}, b_2, b_{\nu_2}, b_3, b_{\nu_3})) \\
&= \phi^{-1}(\sigma_{B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n} \otimes B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}}) \\
&\quad (\text{id}^{\otimes p_1} \otimes \sigma_{B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n}})(\tilde{b}_1(b_{\nu_1} \otimes \tilde{b}_2(b_{\nu_2} \otimes \tilde{b}_3 b_{\nu_3}))) \\
&= \phi^{-1} \sigma_{B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n} \otimes B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}}(\tilde{b}_1(b_{\nu_1} \otimes \tilde{b}_2(b_{\nu_3} \otimes \tilde{b}_3^* b_{\nu_2}))) \\
&= \phi^{-1}(\tilde{b}_1((b_{\nu_3} \otimes \tilde{b}_3^* b_{\nu_2}) \otimes \tilde{b}_2^* b_{\nu_1})) \\
&= Y^{(p_1,p_2)}(b_1, b_{\nu_3}, b_3^*, b_{\nu_2}, b_2^*, b_{\nu_1})
\end{aligned}$$

and

$$\begin{aligned}
& \sigma_{1,p_2,n} \circ \sigma_{1,p_1,p_2}(Y^{(p_1,p_2)}(b^1, b^{\nu^1}, b^2, b^{\nu^2}, b^3, b^{\nu^3})) \\
&= \phi^{-1}(\sigma_{B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}} \otimes B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n}}) \\
&\quad (\sigma_{B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}} \otimes \text{id}^{\otimes (n-p_2)}) \phi(Y^{(p_1,p_2)}(b^1, b^{\nu^1}, b^2, b^{\nu^2}, b^3, b^{\nu^3})) \\
&= \phi^{-1}(\sigma_{B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}} \otimes B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n}}) \\
&\quad (\sigma_{B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_1}+1} \otimes \dots \otimes B_{\lambda_{p_2}}} \otimes \text{id}^{\otimes (n-p_2)})(\tilde{b}^1(b^{\nu^1} \otimes \tilde{b}^2 b^{\nu^2} \otimes \tilde{b}^3 b^{\nu^3})) \\
&= \phi^{-1}(\sigma_{B_{\lambda_1+1} \otimes \dots \otimes B_{\lambda_{p_2}} \otimes B_{\lambda_1} \otimes \dots \otimes B_{\lambda_{p_1}}, B_{\lambda_{p_2}+1} \otimes \dots \otimes B_{\lambda_n}})(\tilde{b}^1((b^{\nu^2} \otimes \widetilde{(b^2)^* b^{\nu^1}}) \otimes \tilde{b}^3 b^{\nu^3})) \\
&= \phi^{-1}(\tilde{b}^1(b^{\nu^3} \otimes \widetilde{(b^3)^* (b^{\nu^2} \otimes \widetilde{(b^2)^* b^{\nu^1}})})) \\
&= Y^{(p_1,p_2)}(b^1, b^{\nu^3}, (b^3)^*, b^{\nu^2}, (b^2)^*, b^{\nu^1}),
\end{aligned}$$

and the result follows.  $\square$

## 5. THE CACTUS RELATION

In this section we use the geometric description of the crystal commutor discussed in Section 4 to show that the commutor satisfies the cactus relation for arbitrary simply-laced Kac-Moody algebras. This result will be extended to symmetrizable Kac-Moody algebras in Section 6

**5.1. An involution of highest weight irreducible components.** In Section 3.2 we described an involution on the set of irreducible components of Lusztig quiver varieties corresponding to Kashiwara's involution. We now discuss a similar involution on Nakajima quiver varieties. Fix Hermitian forms on  $V$  and  $W$  such that the form on  $W$  is compatible with the decomposition  $W = \bigoplus_{i=1}^n W^i$  (that is, vectors in different summands are orthogonal). Consider a point  $(x, s, t) \in \mathbf{M}(V, W)$ . Then  $(x, s, t)^\dagger \stackrel{\text{def}}{=} (x^\dagger, t^\dagger, s^\dagger) \in \mathbf{M}(V, W)$ . Now

$$\mu(x, s, t)_i^\dagger = \left( \sum_{h \in H, \text{in}(h)=i} \varepsilon(h) x_h x_{\bar{h}} + st \right)^\dagger = \sum_{h \in H, \text{in}(h)=i} \varepsilon(h) x_{\bar{h}}^\dagger x_h^\dagger + t^\dagger s^\dagger = \mu(x^\dagger, t^\dagger, s^\dagger)_i$$

and so

$$(x, s, t) \in \mu^{-1}(0) \iff (x, s, t)^\dagger \in \mu^{-1}(0).$$

Recall that  $(x, s, t) \in \mu^{-1}(0)$  is a stable point if the only  $I$ -graded  $x$ -invariant subspace of  $V$  contained in the kernel of  $t$  is zero. We say that  $(x, s, t) \in \mu^{-1}(0)$  is *costable* if the only  $I$ -graded  $x$ -invariant subspace of  $V$  containing the image of  $s$  is  $V$  itself. Then it is easy to see that  $(x, s, t)$  is stable if and only if  $(x, s, t)^\dagger$  is costable.

**Lemma 5.1.** *For  $x$  nilpotent,  $(x, s, t)$  is costable if and only if  $\varepsilon_i(x, s, t) = 0$  for all  $i \in I$ .*

A proof of this lemma appears in [18, Lemma 2.9.4]. We include a proof for completeness.

*Proof.* Suppose that for some  $i \in I$ ,  $\varepsilon_i(x, s, t) > 0$ . Then  $\text{im } \tau_i \subsetneq V_i$ . Define  $V'_j = V_j$  for  $j \neq i$  and  $V'_i = \text{im } \tau_i$ . Then  $V'$  is an  $x$ -invariant proper subspace of  $V$  containing  $\text{im } s$ . Therefore  $(x, s, t)$  is not costable.

Now suppose that  $(x, s, t)$  is not costable. Then there exists a proper  $I$ -graded  $x$ -invariant subspace  $S \subsetneq V$  containing the image of  $s$ . Thus  $S^\perp$  is a nonzero  $I$ -graded  $x^\dagger$ -invariant subspace of  $V$ . Since  $x$  (and hence  $x^\dagger$ ) is nilpotent, we can choose a minimal  $N$  such that  $x_{h_N}^\dagger \cdots x_{h_1}^\dagger|_{S^\perp} = 0$  for all directed paths  $h_1 h_2 \dots h_N$  in our quiver. By the minimality of  $N$ , there exists a directed path  $h_1 h_2 \dots h_{N-1}$  such that  $x_{h_{N-1}}^\dagger \cdots x_{h_1}^\dagger|_{S^\perp}$  is nonzero. Let  $v \in S_i^\perp$ ,  $i = \text{out}(h_{N-1})$ , be a nonzero vector in the image of this map. Now, suppose  $h \in H$  with  $\text{in}(h) = i$ . By our choice of  $N$ ,  $v$  is killed by  $x_h^\dagger$ . Then for all  $u \in \text{out}(h)$ ,

$$\langle x_h(u), v \rangle = \langle u, x_h^\dagger(v) \rangle = \langle u, 0 \rangle = 0.$$

Therefore  $v \in (\text{im } x_h)^\perp$  for all  $h \in H$  with  $\text{in}(h) = i$ . Furthermore, for all  $w \in W$ ,  $\langle s(w), v \rangle = 0$  since  $s(W) \subseteq S$  and  $v \in S^\perp$ . Thus  $0 \neq v \in (\text{im } \tau_i)^\perp$ . Therefore  $\text{im } \tau_i \neq V_i$  and so  $\varepsilon_i(x, s, t) > 0$ .  $\square$

Let  $Y \in B(\mathbf{w}^1, \dots, \mathbf{w}^n)$  be a highest weight element. Then,  $\varepsilon_i(Y) = 0$  for all  $i \in I$ . In other words,  $\varepsilon_i([x, s, t]) = 0$  for all  $i \in I$  and  $[x, s, t]$  in a dense subset  $U$  of  $Y$ . Fix  $[x, s, t] \in U$ . Recall that  $(x, s, t) \in \mu^{-1}(0)$  implies  $(x, s, t)^\dagger \in \mu^{-1}(0)$ . By Proposition 3.6 and Lemma 5.1,  $(x, s, t)$  is costable and thus  $(x, s, t)^\dagger$  is stable. For  $g \in G_V$ ,

$$(g \cdot (x, s, t))^\dagger = (g x g^{-1}, g s, t g^{-1})^\dagger = ((g^{-1})^\dagger x^\dagger g^\dagger, (g^{-1})^\dagger t^\dagger, s^\dagger g^\dagger) = (g^{-1})^\dagger \cdot (x, s, t)^\dagger,$$

and so

$$G_V \cdot (x, s, t)^\dagger = (G_V \cdot (x, s, t))^\dagger \stackrel{\text{def}}{=} \{(x', s', t')^\dagger \mid (x', s', t') \in G_V \cdot (x, s, t)\}.$$

Thus  $[x, s, t]^\dagger = G_V \cdot (x, s, t)^\dagger$  is a well defined point of  $\mathfrak{M}(\mathbf{w})$ . Let

$$\lambda'(z) = \text{id}_{W^n} \oplus z \text{id}_{W^{n-1}} \oplus \cdots \oplus z^{n-1} \text{id}_{W^1} = z^{n-1} \lambda(z^{-1}).$$

Then

$$\begin{aligned} \lambda'(z) * (x, s, t)^\dagger &= \lambda'(z) * (x^\dagger, t^\dagger, s^\dagger) \\ &= (x^\dagger, t^\dagger \lambda'(z)^{-1}, \lambda'(z) s^\dagger) \\ &= (x^\dagger, (\lambda'(\bar{z})^{-1} t)^\dagger, (s \lambda'(\bar{z}))^\dagger) \\ &= (x, s \lambda'(\bar{z}), \lambda'(\bar{z})^{-1} t)^\dagger \\ &= (\lambda'(\bar{z})^{-1} * (x, s, t))^\dagger \end{aligned}$$

$$\begin{aligned}
&= ((\bar{z}^{1-n}\lambda(\bar{z})) * (x, s, t))^{\dagger} \\
&= (\bar{z}^{n-1}\text{id}_V \cdot \lambda(\bar{z}) * (x, s, t))^{\dagger} \\
&= z^{1-n}\text{id}_V \cdot (\lambda(\bar{z}) * (x, s, t))^{\dagger}.
\end{aligned}$$

Therefore

$$\lambda'(z) * [x, s, t]^{\dagger} = (\lambda(\bar{z}) * [x, s, t])^{\dagger}$$

and so

$$\lim_{z \rightarrow 0} \lambda'(z) * \pi([x, s, t]^{\dagger}) = \lim_{z \rightarrow 0} \pi(\lambda'(z) * [x, s, t]^{\dagger}) = \left( \lim_{\bar{z} \rightarrow 0} \pi(\lambda(\bar{z}) * [x, s, t]) \right)^{\dagger} = 0.$$

Thus

$$U^{\dagger} \stackrel{\text{def}}{=} \{[x, s, t]^{\dagger} \mid [x, s, t] \in U\}$$

is a well defined subset of  $\mathfrak{T}(\mathbf{w}^n, \dots, \mathbf{w}^1)$ . Since  $\mathfrak{T}(\mathbf{w}^1, \dots, \mathbf{w}^n)$  and  $\mathfrak{T}(\mathbf{w}^n, \dots, \mathbf{w}^1)$  have the same pure dimension (they are both lagrangian subvarieties of  $\mathfrak{M}(\mathbf{w})$  [19, Prop 3.15]),  $U^{\dagger}$  is a dense subset of some irreducible component of  $\mathfrak{T}(\mathbf{w}^n, \dots, \mathbf{w}^1)$  which we will denote by  $Y^{\dagger}$ .

## 5.2. Proof of the cactus relation.

**Proposition 5.2.** *Suppose  $Y$  is a highest weight element of the crystal  $B(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3)$ . Then*

$$\sigma_{1,1,3} \circ \sigma_{2,2,3}(Y) = \sigma_{1,2,3} \circ \sigma_{1,1,2}(Y) = Y^{\dagger}.$$

*Proof.* Choose  $b \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes B_{\lambda_3}$  such that  $Y = Y_b$ , where  $\lambda_i = \lambda_{\mathbf{w}^i}$  for  $i = 1, 2, 3$ . Then we have

$$\begin{aligned}
b_{(1,2)} &= (b_{\infty}, b_{\lambda_1}, b_2, b_{\lambda_2}, b_3, b_{\lambda_3}), & Y &= Y^{(1,2)}(b_{\infty}, b_{\lambda_1}, b_2, b_{\lambda_2}, b_3, b_{\lambda_3}) \\
b^{(1,2)} &= (b_{\infty}, b_{\lambda_1}, b^2, b_{\lambda_2}, b^3, b_{\lambda_3}), & Y &= Y^{(1,2)}(b_{\infty}, b_{\lambda_1}, b^2, b_{\lambda_2}, b^3, b_{\lambda_3}),
\end{aligned}$$

for some  $b_2, b_3, b^2, b^3$ . Note that it follows from Definition 4.1 that  $\nu^i = \nu_i = \lambda_i$  and  $b^{\lambda_i} = b_{\lambda_i}$  for  $i = 1, 2, 3$ .

By Proposition 4.7, we have

$$\begin{aligned}
\sigma_{1,1,3} \circ \sigma_{2,2,3}(Y) &= Y^{(1,2)}(b_{\infty}, b_{\lambda_3}, b_3^*, b_{\lambda_2}, b_2^*, b_{\lambda_1}), \\
\sigma_{1,2,3} \circ \sigma_{1,1,2}(Y) &= Y^{(1,2)}(b_{\infty}, b_{\lambda_3}, (b^3)^*, b_{\lambda_2}, (b^2)^*, b_{\lambda_1}).
\end{aligned}$$

Recall that  $Y^{(1,2)}(b_{\infty}, b_{\lambda_1}, b_2, b_{\lambda_2}, b_3, b_{\lambda_3})$  is the unique element of  $\mathcal{Y}^{(1,2)}(b_{\infty}, b_{\lambda_1}, b_2, b_{\lambda_2}, b_3, b_{\lambda_3})$ . Thus, by Definition 4.2,  $Y^{(1,2)}(b_{\infty}, b_{\lambda_1}, b_2, b_{\lambda_2}, b_3, b_{\lambda_3})$  is the unique irreducible component of  $\mathfrak{T}(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3)$  such that for all  $[x, s, t]$  in a dense subset of the component, there exists a flag  $0 = V^4 \subseteq V^3 \subseteq V^2 = V^1 = V$  such that

$$\begin{aligned}
(5.1) \quad x(V^i) &\subseteq V^i, \quad s(W^i) \subseteq V^{i+1}, \quad t(V^i) \subseteq W^{i,n}, \quad 1 \leq i \leq 3, \\
x^{V/V^3} &\in X_{b_2}, \quad x^{V^3} \in X_{b_3}.
\end{aligned}$$

Note that  $Y_{b_{\lambda_i}} = \{0\}$  and so  $V^{\nu^i} = V^{i+1}$  for  $i = 1, 2, 3$ . Also  $b_1 = b_{\infty}$  and so  $X_{b_1} = \{0\}$ .

Similarly,  $Y^{(1,2)}(b_{\infty}, b_{\lambda_3}, b_3^*, b_{\lambda_2}, b_2^*, b_{\lambda_1})$  is the unique irreducible component of  $\mathfrak{T}(\mathbf{w}^3, \mathbf{w}^2, \mathbf{w}^1)$  such that for all  $[x', s', t']$  in a dense subset of the component, there exists a flag  $0 = S^4 \subseteq S^3 \subseteq S^2 = S^1 = V$  such that

$$(5.2) \quad x'(S^i) \subseteq S^i, \quad s'(W^{4-i}) \subseteq S^{i+1}, \quad t'(S^i) \subseteq W^{1,4-i}, \quad 1 \leq i \leq 3,$$

$$(x')^{S/S^3} \in X_{b_3^*}, \quad (x')^{S^3} \in X_{b_2^*}.$$

For a point  $(x, s, t)$  with a flag  $0 = V^4 \subseteq V^3 \subseteq V^2 = V^1 = V$  satisfying (5.1), set  $S^1 = S^2 = V$ ,  $S^3 = (V^3)^\perp$ , and  $S^4 = 0$ . We claim that  $(x', s', t') = (x, s, t)^\dagger = (x^\dagger, t^\dagger, s^\dagger)$  satisfies conditions (5.2). First we have  $x^\dagger(S^3) = x^\dagger((V^3)^\perp) \subseteq (V^3)^\perp = S^3$  since  $x(V^3) \subseteq V^3$ . And  $x^\dagger(S^i) \subseteq S^i$  for  $i = 1, 2, 4$  trivially. The condition  $t^\dagger(W^3) \subseteq V = S^2$  is also trivial. We have  $t^\dagger(W^2) \subseteq (V^3)^\perp = S^3$  since  $t(V^3) \subseteq W^3$  and  $t^\dagger(W^1) = 0 = S^4$  since  $t(V) = t(V^2) \subseteq W^{2,3}$ . The condition  $s^\dagger(S^1) \subseteq W = W^{1,3}$  holds trivially,  $s^\dagger(S^2) = s^\dagger(V) \subseteq W^{1,2}$  since  $s(W^3) \subseteq V^4 = 0$ , and  $s^\dagger(S^3) = s^\dagger((V^3)^\perp) \subseteq W^1$  since  $s(W^{2,3}) \subseteq V^3$ . The final two conditions then follow from the fact that  $x \in X_b$  if and only if  $x^\dagger \in X_{b^*}$  (see Section 3.2). Conversely,  $(x, s, t)^\dagger$  satisfies (5.2) only if  $(x, s, t)$  satisfies (5.1). Therefore

$$\sigma_{1,1,3} \circ \sigma_{2,2,3}(Y) = Y^{(1,2)}(b_\infty, b_{\lambda_3}, b_3^*, b_{\lambda_2}, b_2^*, b_{\lambda_1}) = Y^\dagger.$$

An analogous argument shows that

$$\sigma_{1,2,3} \circ \sigma_{1,1,2}(Y) = Y^{(1,2)}(b_\infty, b_{\lambda_3}, (b^3)^*, b_{\lambda_2}, (b^2)^*, b_{\lambda_1}) = Y^\dagger.$$

□

**Corollary 5.3.** *We have*

$$\sigma_{1,1,3} \circ \sigma_{2,2,3} = \sigma_{1,2,3} \circ \sigma_{1,1,2} : B(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3) \rightarrow B(\mathbf{w}^3, \mathbf{w}^2, \mathbf{w}^1).$$

*Proof.* Proposition 5.2 asserts that  $\sigma_{1,1,3} \circ \sigma_{2,2,3} = \sigma_{1,2,3} \circ \sigma_{1,1,2}$  when restricted to highest weight elements. The result then follows from the fact that the maps  $\sigma_{p,q,r}$  are crystal morphisms. □

**Theorem 5.4.** *For a Kac-Moody algebra with symmetric Cartan matrix and dominant integral weights  $\lambda_1, \lambda_2, \lambda_3$ ,*

$$\sigma_{B_{\lambda_1}, B_{\lambda_3} \otimes B_{\lambda_2}} \circ (\text{id} \otimes \sigma_{B_{\lambda_2}, B_{\lambda_3}}) = \sigma_{B_{\lambda_2} \otimes B_{\lambda_1}, B_{\lambda_3}} \circ (\sigma_{B_{\lambda_1}, B_{\lambda_2}} \otimes \text{id}).$$

*That is, the crystal commutor satisfies the cactus relation.*

*Proof.* For  $i = 1, 2, 3$ , choose  $\mathbf{w}^i$  such that  $\lambda_i = \lambda_{\mathbf{w}^i}$ . Then we have the crystal isomorphism

$$\phi : B(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3) \rightarrow B_{\lambda_1} \otimes B_{\lambda_2} \otimes B_{\lambda_3}$$

of Proposition 3.2. The result then follows from Corollary 5.3 and the definition of  $\sigma_{p,q,r}$ . □

## 6. EXTENSION TO SYMMETRIZABLE KAC-MOODY ALGEBRAS

We now extend the results of the previous sections to the more general setting of symmetrizable Kac-Moody algebras, dropping the restriction that the Cartan matrix is symmetric. Our main tool will be a well-known method for obtaining the Cartan matrices, root systems, Dynkin diagrams, etc. of non-simply-laced type from the corresponding objects of simply-laced type via an admissible automorphism or “folding” of a Dynkin diagram. We refer the reader to [4, 13, 21] for details.



**6.1. Admissible automorphisms.** Let  $(I, E)$  be a graph without loops where  $I$  is the set of vertices and  $E$  is the set of edges. We allow multiple edges between pairs of vertices. The corresponding symmetric generalized Cartan matrix is the matrix  $A$  indexed by  $I$  with entries

$$a_{ij} = \begin{cases} 2 & i = j \\ -\#\{\text{edges with endpoints } i \text{ and } j\} & i \neq j \end{cases}.$$

As usual, let  $Q = (I, H)$  be the (double) quiver associated to the graph. That is, for each  $e \in E$ , we have two elements of  $H$  arising from the two possible orientations of  $e$ . Let  $\mathfrak{g}(Q)$  denote the symmetric Kac-Moody algebra associated to the above Cartan matrix, with root system  $\Delta(Q)$  (see [5]).

An *admissible automorphism*  $\mathbf{a}$  of  $Q$  is an automorphism of the underlying graph such that no edge joins two vertices in the same  $\mathbf{a}$ -orbit. Let  $\mathbf{I}$  denote the set of vertex  $\mathbf{a}$ -orbits. Following [13] we construct a symmetric matrix  $M$  indexed by  $\mathbf{I}$ . The  $(\mathbf{i}, \mathbf{j})$  entry of  $M$  is defined to be

$$m_{\mathbf{ij}} = \begin{cases} 2\#\{\text{vertices in } \mathbf{i}\text{th orbit}\} & \mathbf{i} = \mathbf{j} \\ -\#\{\text{edges joining a vertex in } \mathbf{i}\text{th orbit and a vertex in } \mathbf{j}\text{th orbit}\} & \mathbf{i} \neq \mathbf{j} \end{cases}.$$

Then let

$$d_{\mathbf{i}} = m_{\mathbf{ii}}/2 = \#\{\text{vertices in } \mathbf{i}\text{th orbit}\}$$

and set  $D = \text{diag}(d_{\mathbf{i}})$ . Then  $C = D^{-1}M$  is a symmetrizable generalized Cartan matrix. Let  $\Gamma$  denote the corresponding valued graph. That is,  $\Gamma$  has vertex set  $\mathbf{I}$  and whenever  $c_{\mathbf{ij}} \neq 0$ , we draw an edge joining  $\mathbf{i}$  and  $\mathbf{j}$  equipped with the ordered pair  $(|c_{\mathbf{ji}}|, |c_{\mathbf{ij}}|)$ . It is known [13, Prop 14.1.2] that any symmetrizable generalized Cartan matrix (and corresponding valued graph) can be obtained from a pair  $(Q, \mathbf{a})$  in this way. The fact that  $\mathbf{a}$  is admissible ensures that  $\Gamma$  has no vertex loops. Let  $\mathfrak{g}(\Gamma)$  be the Kac-Moody algebra associated to  $C$ , with root system  $\Delta(\Gamma)$ .

Let  $(-, -)_Q$  and  $(-, -)_\Gamma$  be the symmetric bilinear forms determined by the matrices  $A$  and  $M$  respectively. The automorphism  $\mathbf{a}$  acts naturally on the root lattice  $\mathbb{Z}I$  for  $Q$ , and  $(-, -)_Q$  is  $\mathbf{a}$ -invariant. There is a canonical bijection

$$f : (\mathbb{Z}I)^{\mathbf{a}} \rightarrow \mathbb{Z}\mathbf{I}, \quad f(\beta)_{\mathbf{i}} = \beta_i \text{ for any } i \in \mathbf{i},$$

from the fixed points in the root lattice for  $Q$  to the root lattice for  $\Gamma$ . We will often suppress the bijection  $f$  and consider the root lattice of  $\Gamma$  to be the fixed points in the root lattice for  $Q$ . In particular, we have the simple roots for  $\Gamma$  given by

$$\alpha_{\mathbf{i}} = \sum_{i \in \mathbf{i}} \alpha_i.$$

We also define

$$h_{\mathbf{i}} = \frac{1}{d_{\mathbf{i}}} \sum_{i \in \mathbf{i}} h_i.$$

Then the entries of  $C$  are given by  $c_{\mathbf{ij}} = \langle \alpha_{\mathbf{i}}, h_{\mathbf{j}} \rangle$ .

It was shown in [9] (see also [21, Lemma 5.1]) that for vertices  $i$  and  $j$  in the same orbit  $\mathbf{i}$ , we have

$$\tilde{e}_i \tilde{e}_j = \tilde{e}_j \tilde{e}_i, \quad \tilde{f}_i \tilde{f}_j = \tilde{f}_j \tilde{f}_i$$

and for any  $\mathfrak{g}(Q)$ -crystal the operators

$$\tilde{e}_{\mathbf{i}} = \prod_{i \in \mathbf{i}} \tilde{e}_i, \quad \tilde{f}_{\mathbf{i}} = \prod_{i \in \mathbf{i}} \tilde{f}_i$$

are well defined. If  $B_{\infty}^Q$  is the  $\mathfrak{g}(Q)$ -crystal corresponding to the crystal base of  $U_q^-(\mathfrak{g}(Q))$ , then for  $b \in B_{\infty}^Q$ , we also define

$$\varepsilon_{\mathbf{i}}(b) = \max\{k \geq 0 \mid \tilde{e}_{\mathbf{i}}^k b \neq 0\}, \quad \varphi_{\mathbf{i}}(b) = \varepsilon_{\mathbf{i}}(b) + \langle h_{\mathbf{i}}, \text{wt}(b) \rangle.$$

Let  $B_{\infty}^{\Gamma}$  be the subset of  $B_{\infty}^Q$  generated by the  $\tilde{f}_{\mathbf{i}}$ ,  $\mathbf{i} \in \mathbf{I}$  acting on the highest weight element  $b_{\infty} \in B_{\infty}^Q$ . If we restrict the map  $\text{wt} : B_{\infty}^Q \rightarrow P(Q)$ , where  $P(Q)$  is the weight lattice of  $\mathfrak{g}(Q)$ , to the subset  $B_{\infty}^{\Gamma}$ , the image lies in the subset of  $P(Q)$  invariant under the natural action of  $\mathbf{a}$ . We can therefore view it as a map  $\text{wt} : B_{\infty}^{\Gamma} \rightarrow P(\Gamma)$  where  $P(\Gamma)$  is the weight lattice of  $\mathfrak{g}(\Gamma)$ .

**Proposition 6.1.** *The set  $B_{\infty}^{\Gamma}$  along with the maps  $\tilde{e}_{\mathbf{i}}$ ,  $\tilde{f}_{\mathbf{i}}$ ,  $\varepsilon_{\mathbf{i}}$ ,  $\varphi_{\mathbf{i}}$ ,  $\mathbf{i} \in \mathbf{I}$  and  $\text{wt}$  defined above is a  $\mathfrak{g}(\Gamma)$ -crystal isomorphic to the crystal associated to the crystal base of  $U_q^-(\mathfrak{g}(\Gamma))$ .*

*Proof.* This proposition was proven in [9]. See also [21, Prop 5.2, Prop 5.5].  $\square$

Let  $\lambda \in P(Q)^+$  be a dominant integral weight of  $\mathfrak{g}(Q)$  such that  $\mathbf{a}(\lambda) = \lambda$ . Thus we can also think of  $\lambda$  as a dominant integral weight of  $\mathfrak{g}(\Gamma)$ . Let  $B_{\lambda}^Q$  denote the  $\mathfrak{g}(Q)$ -crystal corresponding to the irreducible highest weight representation with highest weight  $\lambda$ . Let  $B_{\lambda}^{\Gamma}$  be the subset of  $B_{\lambda}^Q$  generated by the  $\tilde{f}_{\mathbf{i}}$ ,  $\mathbf{i} \in \mathbf{I}$ , acting on the highest weight element  $b_{\lambda}$  of  $B_{\lambda}^Q$ . If we restrict the map  $\text{wt} : B_{\lambda}^Q \rightarrow P(Q)$  to the subset  $B_{\lambda}^{\Gamma}$ , then the image lies in the subset of  $P(Q)$  that is invariant under the action of  $\mathbf{a}$ . Thus we can view it as a map  $\text{wt} : B_{\lambda}^{\Gamma} \rightarrow P(\Gamma)$ .

**Proposition 6.2.** *The set  $B_{\lambda}^{\Gamma}$  along with the maps  $\tilde{e}_{\mathbf{i}}$ ,  $\tilde{f}_{\mathbf{i}}$ ,  $\varepsilon_{\mathbf{i}}$ ,  $\varphi_{\mathbf{i}}$ ,  $\mathbf{i} \in \mathbf{I}$  and  $\text{wt}$  defined above is a  $\mathfrak{g}(\Gamma)$ -crystal isomorphic to the  $\mathfrak{g}(\Gamma)$ -crystal corresponding to the irreducible highest weight representation of  $U_q(\mathfrak{g}(\Gamma))$  with highest weight  $\lambda$ .*

*Proof.* This proposition was proven in [9]. See also [21, Prop 7.1, Prop 7.4].  $\square$

**6.2. The cactus relation for symmetrizable Kac-Moody algebras.** Recall the definition of Kashiwara's involution  $*$  in Section 1.1. It is easily seen that when  $*$  :  $B_{\infty}^Q \rightarrow B_{\infty}^Q$  is restricted to  $B_{\infty}^{\Gamma} \subseteq B_{\infty}^Q$ , it induces an involution  $*$  :  $B_{\infty}^{\Gamma} \rightarrow B_{\infty}^{\Gamma}$  and that this corresponds to Kashiwara's involution on  $B_{\infty}^{\Gamma}$ , considered as a  $\mathfrak{g}(\Gamma)$ -crystal.

Let  $\lambda, \mu \in P(Q)^+$  be dominant integral weights of  $\mathfrak{g}(Q)$  fixed by  $\mathbf{a}$ . Thus they can also be viewed as dominant integral weights of  $\mathfrak{g}(\Gamma)$ .

**Lemma 6.3.** *Let  $(B_{\lambda} \otimes B_{\mu})^{\Gamma}$  be the  $\mathfrak{g}(\Gamma)$ -subcrystal of  $B_{\lambda} \otimes B_{\mu}$  generated by the highest weight element  $b_{\lambda} \otimes b_{\mu}$ . Then*

$$(B_{\lambda} \otimes B_{\mu})^{\Gamma} = \{b \otimes b' \mid b \in B_{\lambda}^{\Gamma}, b' \in B_{\mu}^{\Gamma}\}.$$

*Proof.* It suffices to show that for all  $\mathbf{i} \in \mathbf{I}$  and  $b \otimes b'$  with  $b \in B_{\lambda}^{\Gamma}$ ,  $b' \in B_{\mu}^{\Gamma}$  and  $\tilde{f}_{\mathbf{i}}(b \otimes b') = b_1 \otimes b'_1$ , we have  $b_1 \in B_{\lambda}^{\Gamma}$  and  $b'_1 \in B_{\mu}^{\Gamma}$ . Since  $b \in B_{\lambda}^{\Gamma}$  and  $b' \in B_{\mu}^{\Gamma}$ , we have

$$\varepsilon_i(b) = \varepsilon_j(b), \quad \varepsilon_i(b') = \varepsilon_j(b'), \quad \varphi_i(b) = \varphi_j(b), \quad \varphi_i(b') = \varphi_j(b'), \quad \text{for } i, j \in \mathbf{i}.$$

It also follows from the results of Section 6.1 that for an element  $a$  of  $B_\lambda^\Gamma$  or  $B_\mu^\Gamma$  and  $i, j \in \mathbf{i}$ ,  $i \neq j$ ,

$$\begin{aligned}\varepsilon_i(a) &= \varepsilon_i(\tilde{e}_j a) \text{ if } \tilde{e}_j a \neq 0, \\ \varepsilon_i(a) &= \varepsilon_i(\tilde{f}_j a) \text{ if } \tilde{f}_j a \neq 0, \\ \varphi_i(a) &= \varphi_i(\tilde{e}_j a) \text{ if } \tilde{e}_j a \neq 0, \text{ and} \\ \varphi_i(a) &= \varphi_i(\tilde{f}_j a) \text{ if } \tilde{f}_j a \neq 0.\end{aligned}$$

It follows from the tensor product rule for crystals that

$$\tilde{f}_{\mathbf{i}}(b \otimes b') = \left( \prod_{i \in \mathbf{i}} \tilde{f}_i \right) (b \otimes b') = \begin{cases} \left( \left( \prod_{i \in \mathbf{i}} \tilde{f}_i \right) b \right) \otimes b', & \text{or} \\ b \otimes \left( \prod_{i \in \mathbf{i}} \tilde{f}_i \right) b' \end{cases}$$

and the result follows.  $\square$

It follows from the above results that the crystal commutor  $\sigma_{B_\lambda^\Gamma, B_\mu^\Gamma}$  is obtained by the restriction of the crystal commutor  $\sigma_{B_\lambda^Q, B_\mu^Q}$  to  $B_\lambda^\Gamma \otimes B_\mu^\Gamma \subseteq B_\lambda^Q \otimes B_\mu^Q$ . We thus have the following theorem.

**Theorem 6.4.** *For a symmetrizable Kac-Moody algebra with dominant integral weights  $\lambda_1, \lambda_2, \lambda_3$ ,*

$$\sigma_{B_{\lambda_1}, B_{\lambda_3} \otimes B_{\lambda_2}} \circ (\text{id} \otimes \sigma_{B_{\lambda_2}, B_{\lambda_3}}) = \sigma_{B_{\lambda_2} \otimes B_{\lambda_1}, B_{\lambda_3}} \circ (\sigma_{B_{\lambda_1}, B_{\lambda_2}} \otimes \text{id}).$$

*That is, the crystal commutor satisfies the cactus relation.*

*Proof.* This follows from Theorem 5.4 and the above remarks.  $\square$

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